DAY THIRTY ONE

Vector Algebra

Learning & Revision for the Day

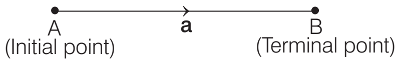
- Vectors and Scalars
- Types of Vectors
- Addition, Subtraction and Scalar Multiplication of Vectors
- Angular Bisectors
- Position Vector (PV)
- Components of a Vector in 2D and 3D
- Scalar (Dot) Product
- Vector (Cross) Product
- Scalar Triple Product
- Vector Triple Product
- Linear Combination, Linear Independence and Dependence

Vectors and Scalars

Physical quantities are divided into two categories **Scalar Quantities** and **Vector Quantities**. Those quantities which have only magnitude and not related to any fixed direction in space are called **scalar quantities** or simply scalars. Examples of scalars are mass, volume, density, work, temperature etc.,

Vectors are those quantities which have both magnitude as well as direction.

Displacement, velocity, acceleration, momentum, weight, force etc., are examples of vector quantities. A directed line-segment is a vector, denoted as $\vec{AB}$ (or **AB**) or simply $\vec{\mathbf{a}}$ (or **a**).



- Magnitude (or length) of a vector **a** is denoted by $|\mathbf{a}|$ and it is always a **non-negative scalar.**

Types of Vectors

(i) A vector whose initial and terminal points coincide is called the **zero** or **null** vector and it is denoted as **0**.

(ii) A vector whose magnitude is 1, is called a **unit vector.** The unit vector in the direction of **a** is given by $\frac{\mathbf{a}}{|\mathbf{a}|}$ and is denoted by $\hat{\mathbf{a}}$. Unit vectors parallel to X-axis, Y-axis and Z-axis are denoted by **i**, **j** and **k**, respectively.

(iii) Vectors are said to be **like** when they have the same sense of direction and **unlike** when they have opposite directions.

(iv) Two vectors **a** and **b** are said to be **equal,** written as $\mathbf{a} = \mathbf{b}$, if they have same length and same direction.

(v) Vectors which are parallel to the same line are called **collinear vectors** or **parallel vector,** otherwise they are called **non-collinear vector**. If **a** and **b** are two collinear vectors, then $\mathbf{a} = \lambda\mathbf{b}$ for same $\lambda \in R$.

(vi) Vectors having the same initial point are called **coinitial vectors.**

(vii) Vectors having the same terminal point are called **coterminous vectors.**

(viii) A system of vectors is said to be **coplanar**, if they are parallel to the same plane or lie in the same plane otherwise they are called **non-coplanar vectors.**

(ix) The vector which has the same magnitude as that of a given vector **a** but opposite direction, is called the **negative** of **a** and is denoted by – **a**.

(x) A vector having the same direction as that of a given vector **a** but magnitude equal to the reciprocal of the given vector is known as the **reciprocal** of **a** and is denoted by $\mathbf{a}^{-1}$.

(xi) A vector which is drawn parallel to a given vector through a specified point in space is called a **localised vector.**

(xii) Vector whose initial points are not specified are called **free vectors.**

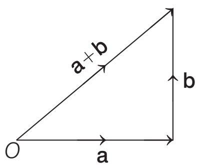
(xiii) When a particle is displaced from point *A* to other point B, then the displacement AB is a vector, called **displacement vector** of the particle.

(xiv) Two vectors are called **orthogonal,** if angle between the two is a right angle.

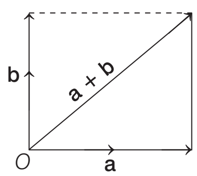
Addition, Subtraction and Scalar Multiplication of Vectors

There are three laws for vector addition, which are given below.

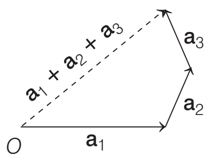
(i) **Triangle law** If the vectors **a** and **b** lie along the two sides of a triangle in consecutive order (as shown in the adjoining figure), then their sum (resultant) $\mathbf{a}+\mathbf{b}$ is represented by the third side, but in opposite direction.



(ii) **Parallelogram law** If the vectors lie along two adjacent sides of a parallelogram (as shown in the adjoining figure), then diagonal of the parallelogram through the common vertex represents their sum.



(iii) **Polygon law** If $(n-1)$ sides of a polygon represents vector

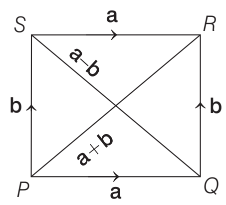


$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots$ in consecutive order, then their sum is represented by the *n*th side, but in opposite direction (as shown in the adjoining figure).

(iv) If $\vec{a}$ and $\vec{b}$ are two vectors, then the **subtraction** of $\vec{b}$ from $\vec{a}$ is defined as the vector sum of $\vec{a}$ and $-\vec{b}$ and it is denoted by $\vec{a}-\vec{b}$, i.e., $\vec{a}-\vec{b} = \vec{a}+(-\vec{b})$.

(v) To subtract **b** from **a**, we reverse the direction of **b** and add it to vector **a**.

(vi) Let *PQRS* be a parallelogram such that $\mathbf{PQ} = \mathbf{a} = \mathbf{SR}$ and $\mathbf{PS} = \mathbf{b} = \mathbf{QR}$. Then, diagonal $\mathbf{PR} = \mathbf{a}+\mathbf{b}$ (addition of vectors) and diagonal $\mathbf{SQ} = \mathbf{a}-\mathbf{b}$ (subtraction of vectors).



(ii) If **a** is a vector and λ is a scalar (i.e. a real number), then λ**a** is a vector whose magnitude is λ times that of **a** and whose direction is the same as that of **a** if $\lambda > 0$ and opposite of **a** if $\lambda < 0$.

Thus, $|\lambda \mathbf{a}| = |\lambda||\mathbf{a}|$

Important Results

(i) $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$

(ii) $\mathbf{a}+(\mathbf{b}+\mathbf{c}) = (\mathbf{a}+\mathbf{b})+\mathbf{c}$

(iii) $\mathbf{a}-\mathbf{b} \neq \mathbf{b}-\mathbf{a}$

(iv) $|\mathbf{a}\pm\mathbf{b}| \le |\mathbf{a}|+|\mathbf{b}|$

(v) $|\mathbf{a}\pm\mathbf{b}| \ge |\mathbf{a}|-|\mathbf{b}|$

(vi) $\lambda_1(\lambda_2\mathbf{a}) = (\lambda_1\lambda_2)\mathbf{a} = \lambda_2(\lambda_1\mathbf{a})$

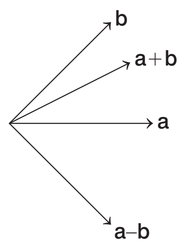
(vii) $(\lambda_1+\lambda_2)\mathbf{a} = \lambda_1\mathbf{a}+\lambda_2\mathbf{a}$

(viii) $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a}+\lambda\mathbf{b}$

NOTE When the sides of a triangle are taken in order, it leads to zero resultant, e.g., In *ABC*, $\mathbf{AB}+\mathbf{BC}+\mathbf{CA} = 0$.

Angular Bisectors

Let **a** and **b** are unit vectors, the internal bisector of angle between **a** and **b** is along $\mathbf{a}+\mathbf{b}$ and external bisector of angle is along $\mathbf{a}-\mathbf{b}$.



If **a** and **b** are not unit vectors, then above angle bisectors are along $\dfrac{\mathbf{a}}{|\mathbf{a}|}+\dfrac{\mathbf{b}}{|\mathbf{b}|}$ and $\dfrac{\mathbf{a}}{|\mathbf{a}|}-\dfrac{\mathbf{b}}{|\mathbf{b}|}$, respectively.

These bisectors are perpendicular to each other.

Position Vector (PV)

Every point $P(x, y, z)$ in space is associated with a vector whose initial point is O(origin) and terminal point is P. This vector is called **position vector** and it is given by

$$\mathbf{OP} \text{ (or } \mathbf{r}) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}.$$

Using distance formula, the magnitude of **OP** (or **r**) is
$|\mathbf{OP}| = \sqrt{x^2 + y^2 + z^2}$

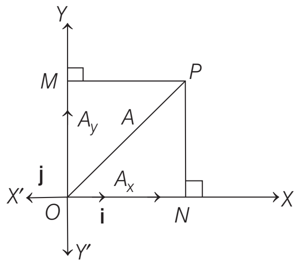
- If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ have position vectors then **a** and **b** respectively, then $\mathbf{AB} = \mathbf{b} - \mathbf{a}$
  and $AB = |\mathbf{b} - \mathbf{a}| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$
- If **a** and **b** are the PV of A and B respectively and **r** be the PV of the point P which divides the join of A and B in the ratio $m : n$, then $\mathbf{r} = \dfrac{m\mathbf{b} \pm n\mathbf{a}}{m \pm n}$.
  Here, '+' sign takes for internal division and '–' sign takes for external division.
- If **a**, **b** and **c** be the PV of three vertices of ΔABC and **r** be the PV of the centroid of ΔABC, then $\mathbf{r} = \dfrac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$

Components of a Vector in 2D and 3D

The process of splitting a vector is called resolution of a vector. The parts of the vector obtained after splitting the vectors are known as the components of the vector.

Let $\mathbf{A}_x$ is the resolved part of **A** along X-axis i.e. the projection of **A** on X-axis. Similarly, $\mathbf{A}_y$ is the resolved part of **A** along Y-axis, i.e. the projection of **A** on Y-axis.

Then, by the parallelogram law,

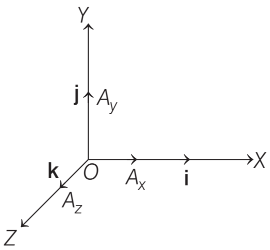


$$\mathbf{OP} = \mathbf{ON} + \mathbf{NP} \quad \text{or} \quad \mathbf{A} = \mathbf{A}_x + \mathbf{A}_y = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}$$

Where A_x and A_y are the magnitudes of $\mathbf{A}_x$ and $\mathbf{A}_y$.

Similarly, in three dimension we can represents vector $\mathbf{A}$.

as $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$



where A_x, A_y and A_z are the magnitudes of $\mathbf{A}_x$, $\mathbf{A}_y$ and $\mathbf{A}_z$, respectively.

Here A_x, A_y and A_z are called **scalar components** of A and $\mathbf{A}_x$, $\mathbf{A}_y$ and $\mathbf{A}_z$ are called **Vector components** of A.

If two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are equal, then their resolved parts will also equal i.e. $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

(i) $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$

(ii) $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}$

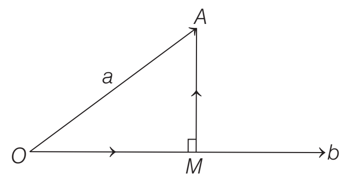
(iii) $\lambda\mathbf{a} = \lambda a_1\mathbf{i} + \lambda a_2\mathbf{j} + \lambda a_3\mathbf{k}$

Scalar (Dot) Product

The scalar product of two vectors **a** and **b** is given by
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta, 0 \le \theta \le \pi$.

Properties of Scalar Product is listed below:

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ [commutative law]
2. $\mathbf{a} \cdot \mathbf{b} = 0$, if $\mathbf{a} \perp \mathbf{b}$
3. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
4. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$
   and $\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j} = 0$
5. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,
   then $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
6. $\mathbf{a} \cdot (\alpha \cdot \mathbf{b}) = \alpha\,(\mathbf{a} \cdot \mathbf{b})$
7. $\mathbf{a} \cdot (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} \pm \mathbf{a} \cdot \mathbf{c}$ [distributive law]
8. For any two vectors **a** and **b**, we have
   (i) $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\,(\mathbf{a} \cdot \mathbf{b})$
   (ii) $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,(\mathbf{a} \cdot \mathbf{b})$
   (iii) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$
   (iv) $\mathbf{a} \cdot \mathbf{b} < 0$ iff **a** and **b** are inclined at an obtuse angle.
   (v) $\mathbf{a} \cdot \mathbf{b} > 0$ iff **a** and **b** are inclined at an acute angle.
9. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are inclined at an angle θ, then
   $$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$
10. If **r** is a vector making angles α, β and γ with OX, OY and OZ respectively, then
    $$\cos\alpha = \mathbf{r} \cdot \mathbf{i}, \cos\beta = \mathbf{r} \cdot \mathbf{j}, \cos\gamma = \mathbf{r} \cdot \mathbf{k}$$
    $$\mathbf{r} = |\mathbf{r}|\cos\alpha\ \mathbf{i} + |\mathbf{r}|\cos\beta\ \mathbf{j} + |\mathbf{r}|\cos\gamma\ \mathbf{k}$$
    If **r** is a unit vector, then
    $$\mathbf{r} = (\cos\alpha)\ \mathbf{i} + (\cos\beta)\ \mathbf{j} + (\cos\gamma)\ \mathbf{k}$$
11. Projection of **a** on **b**(scalar component of **a** along **b**) is



$$OM = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$$

12. Components of $\mathbf{a}$ along and perpendicular to $\mathbf{b}$ are
$$OM = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\cdot\hat{\mathbf{b}} \text{ and } MA = \mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\cdot\hat{\mathbf{b}}, \text{ respectively.}$$

13. **Work done** If a particle acted on by a force $\mathbf{F}$ has displacement $\mathbf{d}$, then work done $= \mathbf{F}\cdot\mathbf{d}$

Vector (Cross) **Product**

The vector product of two vectors $\mathbf{a}$ *and* $\mathbf{b}$ is given by $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ form a right handed system and θ $(0 \le \theta \le \pi)$ is the angle between $\mathbf{a}$ *and* $\mathbf{b}$.

Properties of vector product are listed below:

1. $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$
2. $(\mathbf{a}\times\mathbf{b})^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$
3. $m\mathbf{a}\times\mathbf{b} = m\,(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times m\mathbf{b}$
4. $\mathbf{a}\times(\mathbf{b}\pm\mathbf{c}) = \mathbf{a}\times\mathbf{b}\pm\mathbf{a}\times\mathbf{c}$
   and $(\mathbf{b}\pm\mathbf{c})\times\mathbf{a} = \mathbf{b}\times\mathbf{a}\pm\mathbf{c}\times\mathbf{a}$
5. $\mathbf{a}\times\mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}\,||\,\mathbf{b}$, where, $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.
6. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$
   then, $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
7. The vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{a}\times\mathbf{b}$.
8. The unit vectors perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ are $\pm\dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ and a vector of magnitude λ perpendicular to the plane of ($\mathbf{a}$ and $\mathbf{b}$ or $\mathbf{b}$ and $\mathbf{a}$) is $\dfrac{\lambda\,(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$.
9. If $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are three unit vectors along three mutually perpendicular lines, then
   $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$, $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{i} = -\mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$,
   $\mathbf{k}\times\mathbf{j} = -\mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$, $\mathbf{i}\times\mathbf{k} = -\mathbf{j}$
10. (i) The area of parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$.
    (ii) The area of quadrilateral with diagonals $\mathbf{d}_1$ and $\mathbf{d}_2$ is $\frac{1}{2}|\mathbf{d}_1\times\mathbf{d}_2|$.
    (iii) The area of triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$, is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.
    (iv) If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are position vectors of a ΔABC, then the area $= \frac{1}{2}|(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a})|$.
11. Three points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are collinear if $\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} = \mathbf{0}$

Scalar Triple Product

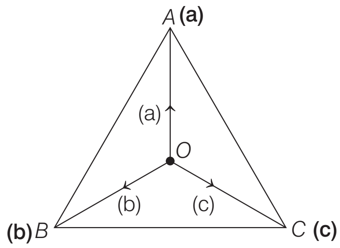
If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three vectors, then their scalar triple product is defined as the dot product of $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$. It is denoted by $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]$. Thus, $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

Properties of Scalar Triple Product are listed below:

1. $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = [\mathbf{b}\;\mathbf{c}\;\mathbf{a}] = [\mathbf{c}\;\mathbf{a}\;\mathbf{b}]$
2. $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = -[\mathbf{b}\;\mathbf{a}\;\mathbf{c}] = -[\mathbf{c}\;\mathbf{b}\;\mathbf{a}] = -[\mathbf{a}\;\mathbf{c}\;\mathbf{b}]$
3. If λ is a scalar, then $[\lambda\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = \lambda\,[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]$
4. $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}_1+\mathbf{c}_2] = [\mathbf{a}\;\mathbf{b}\;\mathbf{c}_1] + [\mathbf{a}\;\mathbf{b}\;\mathbf{c}_2]$
5. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$
6. The scalar triple product of three vectors is zero, if any two of them are equal or parallel or collinear.
7. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar, then $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = 0$
8. If $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = 0$, then any two of the vectors are parallel or $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar or $\mathbf{c} = \alpha\,\mathbf{a} + \beta\mathbf{b}$.
9. Four points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ will be coplanar, if $[\mathbf{b}-\mathbf{a}, \mathbf{c}-\mathbf{a}, \mathbf{d}-\mathbf{a}] = 0$.
10. Volume of parallelopiped, whose coterminous edges are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $|[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]|$.
11. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$
    and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
12. $[\mathbf{a}\times\mathbf{b}\;\;\mathbf{b}\times\mathbf{c}\;\;\mathbf{c}\times\mathbf{a}] = [\mathbf{a}\;\mathbf{b}\;\mathbf{c}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$
13. $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]\cdot[\mathbf{u}\;\mathbf{v}\;\mathbf{w}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{u} & \mathbf{a}\cdot\mathbf{v} & \mathbf{a}\cdot\mathbf{w} \\ \mathbf{b}\cdot\mathbf{u} & \mathbf{b}\cdot\mathbf{v} & \mathbf{b}\cdot\mathbf{w} \\ \mathbf{c}\cdot\mathbf{u} & \mathbf{c}\cdot\mathbf{v} & \mathbf{c}\cdot\mathbf{w} \end{vmatrix}$
14. If $\mathbf{a} = a_1\mathbf{l}+a_2\mathbf{m}+a_3\mathbf{n}$; $\mathbf{b} = b_1\mathbf{l}+b_2\mathbf{m}+b_3\mathbf{n}$ and
    $\mathbf{c} = c_1\mathbf{l}+c_2\mathbf{m}+c_3\mathbf{n}$, then $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\mathbf{l}, \mathbf{m}, \mathbf{n}]$

Tetrahedron and Its Volume

A tetrahedron is a three dimensional figure formed by four triangles, as shown in figure



Volume of tetrahedron
$$OABC = \frac{1}{6}[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]$$

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are position vectors of vertices A, B, C and D of a tetrahedron $ABCD$, then its volume
$$= \frac{1}{6}[\mathbf{a}-\mathbf{d}\;\;\mathbf{b}-\mathbf{d}\;\;\mathbf{c}-\mathbf{d}].$$

Vector Triple Product

If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three vector quantities, then the vectors $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ represents the vector triple product and is given by

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$$
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a}$$

Properties of Vector Triple Product are listed below:

1. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ if some or all $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are zero vectors or $\mathbf{a}$ and $\mathbf{c}$ are collinear.
2. $\mathbf{a} \times \mathbf{b} \times \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = \mathbf{a} \times (\mathbf{b} \times \mathbf{a})$
3. Vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to $\mathbf{a}$ and lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.
4. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$.

Linear Combination, Linear Independence and Dependence

A vector $\mathbf{r}$ is said to be a **linear combination** *of vectors* **a, b, c,** ... etc., if there exist scalars $x, y, z, \ldots$ etc., such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \ldots$

- If two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are **linearly dependent**, then it means
  (i) there are non-zero scalar α and β such that $\alpha\, \mathbf{a} + \beta \mathbf{b} = \mathbf{0}$.
  (ii) $\mathbf{a}$ and $\mathbf{b}$ are parallel.
  (iii) $\mathbf{a}$ and $\mathbf{b}$ are collinear.
  (iv) $\mathbf{a} \times \mathbf{b} = \mathbf{0}$
  Otherwise, $\mathbf{a}$ and $\mathbf{b}$ are **linearly independent**.
- If three non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are **linearly dependent**, then it means
  (i) there are non-zero scalars α, β and γ such that
  $$\alpha\, \mathbf{a} + \beta\, \mathbf{b} + \gamma \mathbf{c} = \mathbf{0}$$
  (ii) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are parallel to same plane.
  (iii) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.
  (iv) $\mathbf{a} = \alpha_1 \mathbf{b} + \alpha_2 \mathbf{c}$ etc.
  (v) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$
  Otherwise $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are **linearly independent**.
- A set of non-zero vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots\ldots, \mathbf{a}_n$ is said to be **linearly independent**, if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = 0$ $\Rightarrow x_1 = x_2 = \ldots = x_n = 0$, where $x_1, x_2, \ldots x_n$ are scalars.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 Let us define the length of a vector $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ as $|a| + |b| + |c|$. This definition coincides with the usual definition of length of a vector $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ iff

(a) $a = b = c = 0$
(b) any two of a, b and c are zero
(c) any one of a, b and c are zero
(d) $a + b + c = 0$

2 The non-zero vectors **a**,**b** and **c** are related by $\mathbf{a} = 8\mathbf{b}$ and $\mathbf{c} = -7\mathbf{b}$. Then, the angle between $\mathbf{a}$ and $\mathbf{c}$ is → **AIEEE 2008**

(a) π (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

3 If a vector $\mathbf{r}$ of magnitude $3\sqrt{6}$ is directed along the bisector of the angle between the vectors $\mathbf{a} = 7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, then $\mathbf{r}$ is equal to

(a) $\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$
(c) $\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$ (d) $\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$

4 If C is the mid-point of AB and P is any point out side AB then → **AIEEE 2005**

(a) $\mathbf{PA} + \mathbf{PB} + \mathbf{PC} = \mathbf{0}$ (b) $\mathbf{PA} + \mathbf{PB} + 2\mathbf{PC} = \mathbf{0}$
(c) $\mathbf{PA} + \mathbf{PB} = \mathbf{PC}$ (d) $\mathbf{PA} + \mathbf{PB} = 2\mathbf{PC}$

5 If the vectors $\mathbf{AB} = 3\mathbf{i} + 4\mathbf{k}$ and $\mathbf{AC} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ are the sides of a ΔABC, then the length of the median through A is → **JEE Mains 2013**

(a) $\sqrt{18}$ (b) $\sqrt{72}$ (c) $\sqrt{33}$ (d) $\sqrt{45}$

6 Let $|\mathbf{a}| = 2\sqrt{2}, |\mathbf{b}| = 3$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{4}$. If a parallelogram is constructed with adjacent sides $2\mathbf{a} - 3\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$, then its longer diagonal is of length

(a) 10 (b) 8
(c) $2\sqrt{26}$ (d) 6

7 If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit vectors, then $|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2$ does not exceed to

(a) 4 (b) 9 (c) 8 (d) 6

8 $\mathbf{a}$ and $\mathbf{c}$ are unit collinear vectors and $|\mathbf{b}| = 6$, then $\mathbf{b} - 3\mathbf{c} = \lambda\mathbf{a}$, if λ is

(a) $-9, 3$ (b) $9, 3$
(c) $3, -3$ (d) None of these

9 If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then what is the angle between $\mathbf{a}$ and $\mathbf{b}$ for $\sqrt{3}\,\mathbf{a} - \mathbf{b}$ to be a unit vector? → **NCERT Exemplar**

(a) 30° (b) 45°
(c) 60° (d) 90°

10 If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors inclined at an angle $\alpha, \alpha \in [0, \pi]$ to each other and $|\mathbf{a} + \mathbf{b}| < 1$. Then, α belong to

(a) $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$ (b) $\left(\frac{2\pi}{3}, \pi\right)$
(c) $\left(0, \frac{\pi}{3}\right)$ (d) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

11 If **a**, **b** and **c** are unit vectors satisfying $\mathbf{a} - \sqrt{3}\mathbf{b} + \mathbf{c} = 0$, then the angle between the vectors **a** and **c** is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

12 The value of a, for which the points, A, B, C with position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $a\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ respectively are the vertices of a right angled triangle with $C = \frac{\pi}{2}$ are → **AIEEE 2006**

(a) −2 and −1 (b) −2 and 1
(c) 2 and −1 (d) 2 and 1

13 If the vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \lambda\mathbf{i} + \mathbf{j} + \mu\mathbf{k}$ are mutually orthogonal, then (λ, μ) is equal to → **AIEEE 2010**

(a) (− 3, 2) (b) (2, − 3)
(c) (− 2, 3) (d) (3, − 2)

14 Let **a** and **b** be two unit vectors. If the vectors $\mathbf{c} = \mathbf{a} + 2\mathbf{b}$ and $\mathbf{d} = 5\mathbf{a} - 4\mathbf{b}$ are perpendicular to each other, then the angle between **a** and **b** is → **AIEEE 2012**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

15 If **p**, **q** and **r** are perpendicular to $\mathbf{q} + \mathbf{r}, \mathbf{r} + \mathbf{p}$ and $\mathbf{p} + \mathbf{q}$ respectively and if $|\mathbf{p} + \mathbf{q}| = 6, |\mathbf{q} + \mathbf{r}| = 4\sqrt{3}$ and $|\mathbf{r} + \mathbf{p}| = 4$, then $|\mathbf{p} + \mathbf{q} + \mathbf{r}|$ is

(a) $5\sqrt{2}$ (b) 10
(c) 5 (d) 15

16 Let **a** and **b** be the position vectors of points A and B with respect to origin and $|\mathbf{a}| = a, |\mathbf{b}| = b$. The points C and D divide AB internally and externally in the ratio 2 : 3. If **OC** and **OD** are perpendicular, then

(a) $9a^2 = 4b^2$ (b) $4a^2 = 9b^2$
(c) $9a = 4b$ (d) $4a = 9b$

17 A vector of magnitude $\sqrt{2}$ coplanar with $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and perpendicular to $\mathbf{i} + \mathbf{j} + \mathbf{k}$ is

(a) $-\mathbf{j} + \mathbf{k}$ (b) $\mathbf{i} - \mathbf{k}$ (c) $\mathbf{i} - \mathbf{j}$ (d) $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

18 If the positive numbers a, b and c are the pth, qth and rth terms of GP, then the vectors $\log a \cdot \mathbf{i} + \log b \cdot \mathbf{j} + \log c \cdot \mathbf{k}$ and $(q - r)\mathbf{i} + (r - p)\mathbf{j} + (p - q)\mathbf{k}$ are

(a) equal (b) parallel
(c) perpendicular (d) None of these

19 The distance of the point B with position vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ from the line passing through the point A, whose position vector is $4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and parallel to the vector $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ is

(a) $\sqrt{10}$ (b) $\sqrt{5}$ (c) $\sqrt{6}$ (d) $\sqrt{8}$

20 Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ be three vectors. A vector of the type $\mathbf{b} + \lambda\mathbf{c}$ for some scalar λ, whose projection on **a** is of magnitude $\sqrt{\frac{2}{3}}$, is

(a) $2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ (b) $2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$
(c) $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ (d) $2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

21 Let **u**, **v** and **w** be three vectors such that

$$|\mathbf{u}| = 1, |\mathbf{v}| = 2, |\mathbf{w}| = 3$$

If the projection of **v** along **u** is equal to that of **w** along **u** and **v** and **w** are perpendicular to each other, then $|\mathbf{u} - \mathbf{v} + \mathbf{w}|$ is equal to

(a) 4 (b) $\sqrt{7}$ (c) $\sqrt{14}$ (d) 2

22 If **a**, **b** and **c** are three mutually perpendicular vectors, then the projection of the vector $l\frac{\mathbf{a}}{|\mathbf{a}|} + m\frac{\mathbf{b}}{|\mathbf{b}|} + n\frac{(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}$ along the angle bisector of the vector **a** and **b** is

(a) $\frac{l^2 + m^2}{\sqrt{l^2 + m^2 + n^2}}$ (b) $\sqrt{l^2 + m^2 + n^2}$

(c) $\frac{\sqrt{l^2 + m^2}}{\sqrt{l^2 + m^2 + n^2}}$ (d) $\frac{l + m}{\sqrt{2}}$

23 Resolved part of vector **a** along the vector **b** is $\mathbf{a}_1$ and that perpendicular to **b** is $\mathbf{a}_2$, then $\mathbf{a}_1 \times \mathbf{a}_2$ is equal to

(a) $\frac{(\mathbf{a} \times \mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2}$ (b) $\frac{(\mathbf{a} \times \mathbf{b})\,\mathbf{a}}{|\mathbf{a}|^2}$

(c) $\frac{(\mathbf{a} \cdot \mathbf{b})\,(\mathbf{b} \times \mathbf{a})}{|\mathbf{b}|^2}$ (d) $\frac{(\mathbf{a} \cdot \mathbf{b})\,(\mathbf{b} \times \mathbf{a})}{|\mathbf{b} \times \mathbf{a}|}$

24 A particle is acted upon by constant forces $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ which displace it from a point $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ to the point $5\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ The work done in standard units by the forces is given by → **AIEEE 2004**

(a) 40 units (b) 30 units
(c) 25 units (d) 15 units

25 If **u** and **v** are unit vectors and θ is the acute angle between them, then $2\mathbf{u} \times 3\mathbf{v}$ is a unit vector for → **AIEEE 2007**

(a) exactly two values of θ
(b) more than two values of θ
(c) no value of θ
(d) exactly one value of θ

26 If the vectors $\mathbf{c}, \mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{b} = \mathbf{j}$ are such that **a**, **c** and **b** form a right handed system, then **c** is → **AIEEE 2002**

(a) $z\mathbf{i} - x\mathbf{k}$ (b) **0**
(c) $y\mathbf{j}$ (d) $-z\mathbf{i} + x\mathbf{k}$

27 If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$, then a vector c such that $\mathbf{a} \times \mathbf{c} = \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c} = 3$ is → **NCERT Exemplar**

(a) $\frac{5}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ (b) $\frac{2}{3}\mathbf{i} + \frac{5}{3}\mathbf{j} + \frac{5}{3}\mathbf{k}$

(c) $\frac{5}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ (d) None of these

28 Vectors **a** and **b** are not perpendicular and **c** and **d** are two vectors satisfying $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ and $\mathbf{a} \cdot \mathbf{d} = 0$. Then, the vector **d** is equal to → **AIEEE 2011**

(a) $\mathbf{c} + \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (b) $\mathbf{b} + \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$

(c) $\mathbf{c} - \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (d) $\mathbf{b} - \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$

29 If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are non-coplanar vectors and p, q are real numbers, then the equality
$[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}]-[p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}]-[2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}]=0$
holds for **➜ AIEEE 2009**
(a) exactly two values of (p, q)
(b) more than two but not all values of (p, q)
(c) all values of (p, q)
(d) exactly one value of (p, q)

30 Let $\mathbf{v}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{w}=\mathbf{i}+3\mathbf{k}$. If $\mathbf{u}$ is a unit vector, then the maximum value of $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ is
(a) -1 (b) $\sqrt{10}+\sqrt{6}$
(c) $\sqrt{59}$ (d) $\sqrt{60}$

31 If $\mathbf{a}=\mathbf{i}-\mathbf{j}$, $\mathbf{b}=\mathbf{j}-\mathbf{k}$, $\mathbf{c}=\mathbf{k}-\mathbf{i}$ and $\mathbf{d}$ is a unit vector such that $\mathbf{a}\cdot\mathbf{d}=0=[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$, then $\mathbf{d}$ is/are
(a) $\pm\frac{\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{3}}$ (b) $\pm\frac{\mathbf{i}+\mathbf{j}-2\mathbf{k}}{\sqrt{6}}$
(c) $\pm\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$ (d) $\pm\mathbf{k}$

32 The vector $(\mathbf{i}\times\mathbf{a}\cdot\mathbf{b})\mathbf{i}+(\mathbf{j}\times\mathbf{a}\cdot\mathbf{b})\mathbf{j}+(\mathbf{k}\times\mathbf{a}\cdot\mathbf{b})\mathbf{k}$ is equal to
(a) $\mathbf{b}\times\mathbf{a}$ (b) $\mathbf{a}$ (c) $\mathbf{a}\times\mathbf{b}$ (d) $\mathbf{b}$

33 The points with position vectors $\alpha\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{i}-\mathbf{j}-\mathbf{k}$, $\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{i}+\mathbf{j}+\beta\mathbf{k}$ are coplanar if
(a) $(1-\alpha)(1+\beta)=0$ (b) $(1-\alpha)(1-\beta)=0$
(c) $(1+\alpha)(1+\beta)=0$ (d) $(1+\alpha)(1-\beta)=0$

34 Let $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{c}=x\mathbf{i}+(x-2)\mathbf{j}-\mathbf{k}$ If the vector $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, then x equal to **➜ AIEEE 2007**
(a) 0 (b) 1
(c) -4 (d) -2

35 If the vectors $p\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{i}+q\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+\mathbf{j}+r\mathbf{k}$ $(p\neq q\neq r\neq 1)$ are coplanar, then the value of $pqr-(p+q+r)$ is **➜ AIEEE 2011**
(a) -2 (b) 2
(c) 0 (d) -1

36 Let $\alpha=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, $\beta=b\mathbf{i}+c\mathbf{j}+a\mathbf{k}$ and $\gamma=c\mathbf{i}+a\mathbf{j}+b\mathbf{k}$ be three coplanar vectors with $a\neq b$ and $\mathbf{v}=\mathbf{i}+\mathbf{j}+\mathbf{k}$. Then, $\mathbf{v}$ is perpendicular to
(a) α (b) β (c) γ (d) All of these

37 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors and λ is a real number, then $[\lambda(\mathbf{a}+\mathbf{b})\ \ \lambda^2\mathbf{b}\ \ \lambda\mathbf{c}]=[\mathbf{a}\ \ \mathbf{b}+\mathbf{c}\ \ \mathbf{b}]$ for
(a) exactly two values of λ **➜ AIEEE 2005**
(b) exactly three values of λ
(c) no value of λ
(d) exactly one value of λ

38 Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors which are pairwise non-collinear. If $\mathbf{a}+3\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b}+2\mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a}+3\mathbf{b}+6\mathbf{c}$ is equal to
(a) a + c (b) a (c) c (d) 0

39 If $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}]=\lambda[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$, then λ is equal to **➜ JEE Mains 2014**
(a) 0 (b) 1
(c) 2 (d) 3

40 If V is the volume of the parallelopiped having three coterminous edges, as a, b and c, then the volume of the parallelopiped having three coterminous edges as

$$\alpha=(\mathbf{a}\cdot\mathbf{a})\mathbf{a}+(\mathbf{a}\cdot\mathbf{b})\mathbf{b}+(\mathbf{a}\cdot\mathbf{c})\mathbf{c}$$
$$\beta=(\mathbf{a}\cdot\mathbf{b})\mathbf{a}+(\mathbf{b}\cdot\mathbf{b})\mathbf{b}+(\mathbf{b}\cdot\mathbf{c})\mathbf{c}$$
$$\gamma=(\mathbf{a}\cdot\mathbf{c})\mathbf{a}+(\mathbf{b}\cdot\mathbf{c})\mathbf{b}+(\mathbf{c}\cdot\mathbf{c})\mathbf{c}$$

(a) V^3 (b) $3V$
(c) V^2 (d) $2V$

41 If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\mathbf{a}+2\mathbf{b}+3\mathbf{c}$, $\lambda\mathbf{b}+4\mathbf{c}$ and $(2\lambda-1)\mathbf{c}$ are non-coplanar for
(a) no value of λ
(b) all except one value of λ
(c) all except two values of λ
(d) all values of λ

42 If $\mathbf{a}=\frac{1}{\sqrt{10}}(3\mathbf{i}+\mathbf{k})$ and $\mathbf{b}=\frac{1}{7}(2\mathbf{i}+3\mathbf{j}-6\mathbf{k})$, then the value of $(2\mathbf{a}-\mathbf{b})\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}+2\mathbf{b})]$ is **➜ AIEEE 2011**
(a) -3 (b) 5
(c) 3 (d) -5

43 Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be three unit vectors such that $\hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}})=\frac{\sqrt{3}}{2}(\hat{\mathbf{b}}+\hat{\mathbf{c}})$. If $\hat{\mathbf{b}}$ is not parallel to $\hat{\mathbf{c}}$, then the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is **➜ JEE Mains 2016**
(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

44 If $\mathbf{a}=\mathbf{j}-\mathbf{k}$ and $\mathbf{c}=\mathbf{i}-\mathbf{j}-\mathbf{k}$. Then, the vector $\mathbf{b}$ satisfying $\mathbf{a}\times\mathbf{b}+\mathbf{c}=0$ and $\mathbf{a}\cdot\mathbf{b}=3$, is **➜ AIEEE 2010**
(a) $-\mathbf{i}+\mathbf{j}-2\mathbf{k}$ (b) $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$
(c) $\mathbf{i}-\mathbf{j}-2\mathbf{k}$ (d) $\mathbf{i}+\mathbf{j}-2\mathbf{k}$

45 Let $\mathbf{u}$ be a vector coplanar with the vectors $\mathbf{a}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$. If $\mathbf{u}$ is perpendicular to $\mathbf{a}$ and $\mathbf{u}\cdot\mathbf{b}=24$, then $|\mathbf{u}|^2$ is equal to **➜ JEE Mains 2018**
(a) 336 (b) 315 (c) 256 (d) 84

46 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that $\mathbf{a}$ is perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$. If the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{3}$, then $|\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}|^2$ is equal to
(a) $\frac{1}{3}$ (b) $\frac{1}{2}$
(c) 1 (d) 2

47 The vectors $\mathbf{a}$ and $\mathbf{b}$ are non-collinear. The value of x, for which the vectors, $\mathbf{c}=(2x+3)\mathbf{a}+\mathbf{b}$ and $\mathbf{d}=(2x+3)\mathbf{a}-\mathbf{b}$ are collinear is
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$
(c) $-\frac{3}{2}$ (d) None of these

48 It is given that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular vectors of equal magnitudes.
Statement I Vector $(\mathbf{a}+\mathbf{b}+\mathbf{c})$ is equally inclined to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

Statement II If α, β and γ are the angles at which $(\mathbf{a}+\mathbf{b}+\mathbf{c})$ is inclined to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, then $\alpha = \beta = \gamma$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

49 **Statement I** For $a = -\frac{1}{\sqrt{3}}$ the volume of the parallelopiped formed by vectors $\mathbf{i} + a\,\mathbf{j}$, $a\,\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{j} + a\,\mathbf{k}$ is maximum.

Statement II The volume of the parallelopiped having three coterminous edges a, b and c is $|[\mathrm{a\ b\ c}]|$.

(a) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(b) Statement I is true; Statement II is false
(c) Statement I is false; Statement II is true
(d) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

50 **Statement I** A relation between the vectors **r**, **a** and **b** is $\mathbf{r} \times \mathbf{a} = \mathbf{b} \Rightarrow \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$.

Statement II $\mathbf{r} \cdot \mathbf{a} = 0$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 Let $\mathbf{a} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\mathbf{b} = \hat{i} + \hat{j}$ and **c** be a vector such that $|\mathbf{c} - \mathbf{a}| = 3$, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$ and the angle between $\mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ is 30°. Then, $\mathbf{a} \cdot \mathbf{c}$ is equal to → **JEE Mains 2017**

(a) $\frac{25}{8}$ (b) 2 (c) 5 (d) $\frac{1}{8}$

2 Given, two vectors are $\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + 2\mathbf{j}$, the unit vector coplanar with the two vectors and perpendicular to first is

(a) $\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$ (b) $\frac{1}{\sqrt{5}}(2\mathbf{i} + \mathbf{j})$
(c) $\pm\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$ (d) None of these

3 Vectors **a** and **b** are such that $|\mathbf{a}| = 1, |\mathbf{b}| = 4$ and $\mathbf{a} \cdot \mathbf{b} = 2$. If $\mathbf{c} = 2\mathbf{a} \times \mathbf{b} - 3\mathbf{b}$, then the angle between **b** and **c** is

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

4 A unit vector **d** is equally inclined at an angle α with the vectors $\mathbf{a} = \cos\theta \cdot \mathbf{i} + \sin\theta \cdot \mathbf{j}$, $\mathbf{b} = -\sin\theta \cdot \mathbf{i} + \cos\theta \cdot \mathbf{j}$ and $\mathbf{c} = \mathbf{k}$. Then, α is equal to

(a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$
(c) $\cos^{-1}\frac{1}{3}$ (d) $\frac{\pi}{2}$

5 Let $\mathbf{p} = 3ax^2\,\mathbf{i} - 2(x-1)\,\mathbf{j}$, $\mathbf{q} = b(x-1)\,\mathbf{i} + x\,\mathbf{j}$ and $ab < 0$. Then, **p** and **q** are parallel for atleast one x in

(a) (0, 1) (b) (1, 0)
(c) (1, 2) (d) None of these

6 If $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = \cos\theta$, then the maximum value of θ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{2\pi}{5}$

7 If $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$ and **c** be two vectors perpendicular to each other in the xy-plane. Then, a vector in the same plane having projections 1 and 2 along **b** and **c** respectively, is

(a) $\mathbf{i} + 2\mathbf{j}$ (b) $2\mathbf{i} - \mathbf{j}$
(c) $2\mathbf{i} + \mathbf{j}$ (d) None of these

8 The unit vector which is orthogonal to the vector $3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ and is coplanar with the vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$, is

(a) $\frac{2\mathbf{i} - 6\mathbf{j} + \mathbf{k}}{\sqrt{41}}$ (b) $\frac{2\mathbf{i} - 3\mathbf{j}}{\sqrt{13}}$
(c) $\frac{3\mathbf{j} - \mathbf{k}}{\sqrt{10}}$ (d) $\frac{4\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}}{\sqrt{34}}$

9 The values of x for which the angle between the vectors $2x^2\,\mathbf{i} + 4x\,\mathbf{j} + \mathbf{k}$ and $7\mathbf{i} - 2\mathbf{j} + x\,\mathbf{k}$ is obtuse and the angle between the Z-axis and $7\mathbf{i} - 2\mathbf{j} + x\,\mathbf{k}$ is acute and less than $\frac{\pi}{6}$ is given by

(a) $0 < x < \frac{1}{2}$ (b) $x > \frac{1}{2}$ or $x < 0$
(c) $\frac{1}{2} < x < 15$ (d) No such value for x

10 If **a** and **b** are unit vectors, then the greatest value of $|\mathbf{a} + \mathbf{b}| + |\mathbf{a} - \mathbf{b}|$ is

(a) 2 (b) 4
(c) $2\sqrt{2}$ (d) $\sqrt{2}$

11 Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{j}$. If **n** is a unit vector such that $\mathbf{u} \cdot \mathbf{n} = 0$ and $\mathbf{v} \cdot \mathbf{n} = 0$, then $|\mathbf{w} \cdot \mathbf{n}|$ is equal to

(a) 3 (b) 0
(c) 1 (d) 2

12 Let **b** and **c** be non-collinear vectors. If **a** is a vector such that $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=4$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(x^2-2x+6)\,\mathbf{b}$ $+\sin y\cdot\mathbf{c}$ then (x, y) lies on the line

(a) $x+y=0$ (b) $x-y=0$
(c) $x=1$ (d) $y=\pi$

13 If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$, $|\mathbf{c}|=1$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=0$, then

$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to

(a) 0 (b) 1 (c) $|\mathbf{a}|^2\,|\mathbf{b}|^2$ (d) $|\mathbf{a}\times\mathbf{b}|^2$

14 If **a** is a unit vector and projection of x along **a** is 2 and $\mathbf{a}\times\mathbf{r}+\mathbf{b}=\mathbf{r}$, then **r** is equal to

(a) $\frac{1}{2}[\mathbf{a}-\mathbf{b}+\mathbf{a}\times\mathbf{b}]$ (b) $\frac{1}{2}[2\mathbf{a}+\mathbf{b}+\mathbf{a}\times\mathbf{b}]$
(c) $\mathbf{a}+\mathbf{a}\times\mathbf{b}$ (d) $\mathbf{a}-\mathbf{a}\times\mathbf{b}$

15 Let **a**, **b** and **c** be unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$. Which one of the following is correct?

(a) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}=0$
(b) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}\neq 0$
(c) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{c}=0$
(d) $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\times\mathbf{a}$ are mutually perpendicular

16 Let G_1, G_2 and G_3 be the centroids of the triangular faces *OBC*, *OCA* and *OAB* of a tetrahedron *OABC*. If V_1 denote the volume of the tetrahedron *OABC* and V_2 that of the parallelopiped with OG_1, OG_2 and OG_3 as three concurrent edges, then

(a) $4V_1=9V_2$ (b) $9V_1=4V_2$
(c) $3V_1=2V_2$ (d) $3V_2=2V_1$

17 *ABC* is triangle, right angled at *A*. The resultant of the forces acting along **AB** and **AC** with magnitudes $\frac{1}{AB}$ and $\frac{1}{AC}$ respectively is the force along **AD**, where *D* is the foot of the perpendicular from *A* onto *BC*. The magnitude of the resultant is

(a) $\frac{(AB)\,(AC)}{AB+AC}$ (b) $\frac{1}{AB}+\frac{1}{AC}$
(c) $\frac{1}{AD}$ (d) $\frac{AB^2+AC^2}{(AB)^2\,(AC)^2}$

18 Let **a**, **b** and **c** be three non-zero vectors such that no two of them are collinear and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is the angle between vectors **b** and **c**, then a value of $\sin\theta$ is **➔ JEE Mains 2015**

(a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{-\sqrt{2}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{-2\sqrt{3}}{3}$

19. Let *ABCD* be a parallelogram such that $\mathbf{AB}=\mathbf{q}$, $\mathbf{AD}=\mathbf{p}$, and $\angle BAD$ be an acute angle. If **r** is the vector that coincides with the altitude directed from the vertex *B* to the side *AD*, then **r** is given by

(a) $\mathbf{r}=3\mathbf{q}-\frac{3(\mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}\cdot\mathbf{q})}\mathbf{p}$ (b) $\mathbf{r}=-\mathbf{q}+\left(\frac{\mathbf{q}\cdot\mathbf{p}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$
(c) $\mathbf{r}=\mathbf{q}-\left(\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{q}}\right)\mathbf{p}$ (d) $\mathbf{r}=-3\mathbf{q}+\frac{3(\mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}\cdot\mathbf{q})}\mathbf{p}$

20 **Statement I** If **u** and **v** are unit vectors inclined at an angle α and *x* is a unit vector bisecting the angle between them, then $x=\frac{\mathbf{u}+\mathbf{v}}{2\cos\frac{\alpha}{2}}$.

Statement II If ΔABC is an isosceles triangle with $AB=AC=1$, then vector representing bisector of angle *A* is given by $\mathbf{AD}=\frac{\mathbf{AB}+\mathbf{AC}}{2}$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

ANSWERS

SESSION 1

1 (b)	**2** (a)	**3** (a)	**4** (d)	**5** (c)	**6** (c)	**7** (b)	**8** (a)	**9** (a)	**10** (b)
11 (b)	**12** (d)	**13** (a)	**14** (c)	**15** (a)	**16** (a)	**17** (a)	**18** (c)	**19** (a)	**20** (b)
21 (c)	**22** (d)	**23** (c)	**24** (a)	**25** (d)	**26** (a)	**27** (a)	**28** (c)	**29** (d)	**30** (c)
31 (b)	**32** (c)	**33** (a)	**34** (d)	**35** (a)	**36** (d)	**37** (c)	**38** (d)	**39** (b)	**40** (a)
41 (c)	**42** (d)	**43** (d)	**44** (a)	**45** (a)	**46** (c)	**47** (c)	**48** (a)	**49** (c)	**50** (b)

SESSION 2

1 (b)	**2** (a)	**3** (b)	**4** (b)	**5** (a)	**6** (c)	**7** (b)	**8** (c)	**9** (d)	**10** (c)
11 (a)	**12** (c)	**13** (d)	**14** (b)	**15** (b)	**16** (a)	**17** (c)	**18** (a)	**19** (b)	**20** (a)

Hints and Explanations

SESSION 1

1 We have,

$|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}| = |a| + |b| + |c|$

$\Rightarrow \sqrt{a^2 + b^2 + c^2} = |a| + |b| + |c|$

$\Rightarrow a^2 + b^2 + c^2 = a^2 + b^2 + c^2 + 2[|a||b| + |b||c| + |c||a|]$

$\Rightarrow |a||b| + |b||c| + |c||a| = 0$

$\Rightarrow ab = bc = ca = 0$

Hence, any two of a, b and c are zero.

2 Since, $\mathbf{a} = 8\mathbf{b}$ and $\mathbf{c} = -7\mathbf{b}$

So, $\mathbf{a}$ is parallel to $\mathbf{b}$ and $\mathbf{c}$ is anti-parallel to $\mathbf{b}$.

$\Rightarrow$ $\mathbf{a}$ and $\mathbf{c}$ are anti-parallel.

So, the angle between $\mathbf{a}$ and $\mathbf{c}$ is π.

3 The required vector

$\mathbf{r} = \lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$, where λ is a scalar.

$\Rightarrow \mathbf{r} = \lambda\left(\frac{1}{9}(7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}) + \frac{1}{3}(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})\right)$

$\Rightarrow \mathbf{r} = \frac{\lambda}{9}(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$

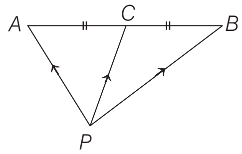
Given, $|\mathbf{r}|^2 = 54$

$\Rightarrow \frac{\lambda^2}{81}(1 + 49 + 4) = 54 \Rightarrow \lambda = \pm 9$

Thus, the required vector is

$\mathbf{r} = \pm(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$.

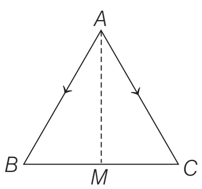
4 Let P be the origin outside of AB and C is mid-point of AB, then



$\mathbf{PC} = \frac{\mathbf{PA} + \mathbf{PB}}{2}$

$\Rightarrow 2\mathbf{PC} = \mathbf{PA} + \mathbf{PB}$

5 We know that, the sum of three vectors of a triangle is zero.



$\therefore \quad \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$

$\Rightarrow \quad \mathbf{BC} = \mathbf{AC} - \mathbf{AB}$

$\Rightarrow \quad \mathbf{BM} = \frac{\mathbf{AC} - \mathbf{AB}}{2}$

[since, M is a mid-point of BC]

Also, $\mathbf{AB} + \mathbf{BM} + \mathbf{MA} = 0$

[by properties of a triangle]

$\Rightarrow \quad \mathbf{AB} + \frac{\mathbf{AC} - \mathbf{AB}}{2} = \mathbf{AM}$

$\Rightarrow \quad \mathbf{AM} = \frac{\mathbf{AB} + \mathbf{AC}}{2}$

$= \frac{3\mathbf{i} + 4\mathbf{k} + 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}}{2}$

$= 4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

$\therefore \quad |\mathbf{AM}| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$

6 We have, $\mathbf{a}\cdot\mathbf{b} = 2\sqrt{2}\cdot 3\cdot\frac{1}{\sqrt{2}} = 6$

The diagonals are $(2\mathbf{a} - 3\mathbf{b}) \pm (\mathbf{a} + \mathbf{b})$

i.e. $3\mathbf{a} - 2\mathbf{b}$ and $\mathbf{a} - 4\mathbf{b}$

$\therefore$ Length of diagonals are

$|3\mathbf{a} - 2\mathbf{b}|^2 = 9|\mathbf{a}|^2 + 4|\mathbf{b}|^2 - 12\mathbf{a}\cdot\mathbf{b}$

$= 9\cdot 8 + 4\cdot 9 - 12\cdot 6 = 36$

and $|\mathbf{a} - 4\mathbf{b}|^2 = |\mathbf{a}|^2 + 16|\mathbf{b}|^2 - 8\mathbf{a}\cdot\mathbf{b}$

$= 8 + 16\cdot 9 - 8\cdot 6 = 104$

So, the length of the longer diagonal is $\sqrt{104}$ i.e. $2\sqrt{26}$.

7 Since, $(\mathbf{a} + \mathbf{b} + \mathbf{c})^2 \geq 0 \Rightarrow 3 + 2\sum \mathbf{a}\cdot\mathbf{b} \geq 0$

or $\quad -2\sum \mathbf{a}\cdot\mathbf{b} \leq 3$

$\therefore \quad |\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 = 6 - 2\sum \mathbf{a}\cdot\mathbf{b} \leq 9$

8 We have, $\mathbf{b} - 3\mathbf{c} = \lambda\mathbf{a}$

Taking scalar product with $\mathbf{c}$, we have

$(\mathbf{b} - 3\mathbf{c})\cdot\mathbf{c} = \lambda(\mathbf{a}\cdot\mathbf{c})$

$\Rightarrow \quad \mathbf{b}\cdot\mathbf{c} - 3(\mathbf{c}\cdot\mathbf{c}) = \lambda(\mathbf{a}\cdot\mathbf{a})$

[$\because |\mathbf{a}| = |\mathbf{c}| = 1$ and $\mathbf{a}$ and $\mathbf{c}$ are collinear vectors]

$\Rightarrow \quad \mathbf{b}\cdot\mathbf{c} - 3 = \lambda$

$\Rightarrow \quad \mathbf{b}\cdot\mathbf{c} = 3 + \lambda \quad$ ...(i)

Again, $\quad \mathbf{b} - 3\mathbf{c} = \lambda\mathbf{a}$

$\Rightarrow \quad |\mathbf{b} - 3\mathbf{c}| = |\lambda\mathbf{a}|$

$\Rightarrow \quad |\mathbf{b} - 3\mathbf{c}|^2 = \lambda^2|\mathbf{a}|^2$

$\Rightarrow \quad |\mathbf{b}|^2 + 9|\mathbf{c}|^2 - 6(\mathbf{b}\cdot\mathbf{c}) = \lambda^2|\mathbf{a}|^2$

$\Rightarrow \quad 36 + 9 - 6(3 + \lambda) = \lambda^2$

[from Eq. (i)]

$\Rightarrow \quad 27 - 6\lambda = \lambda^2 \Rightarrow \lambda^2 + 6\lambda - 27 = 0$

$\therefore \quad \lambda = -9, 3$

9 We have, $(\sqrt{3}\mathbf{a} - \mathbf{b})^2$

$= 3\mathbf{a}^2 + \mathbf{b}^2 - 2\sqrt{3}\mathbf{a}\cdot\mathbf{b} = 1$

$3 + 1 - 2\sqrt{3}\mathbf{a}\cdot\mathbf{b} = 1$

[since, $\mathbf{a}$ and $\mathbf{b}$ are unit vectors]

Thus, $3 = 2\sqrt{3}\mathbf{a}\cdot\mathbf{b}$

$\mathbf{a}\cdot\mathbf{b} = \frac{\sqrt{3}}{2} \Rightarrow |\mathbf{a}||\mathbf{b}|\cos\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \cos\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta = 30° \quad [\because |\mathbf{a}| = |\mathbf{b}| = 1]$

10 Since, $|\mathbf{a} + \mathbf{b}|^2 < 1$

$\Rightarrow \quad 2 + 2\cos\alpha < 1$

$\Rightarrow \quad 4\cos^2\frac{\alpha}{2} < 1$

$\Rightarrow \quad \cos\frac{\alpha}{2} < \frac{1}{2} \Rightarrow \alpha \in \left(\frac{2\pi}{3}, \pi\right)$

11 $\sqrt{3}\mathbf{b} = (\mathbf{a} + \mathbf{c})$

$\Rightarrow \quad 3|\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{c}|^2 + 2\mathbf{a}\cdot\mathbf{c}$

$\Rightarrow \quad 3(1) = 1 + 1 + 2\mathbf{a}\cdot\mathbf{c}$

$\Rightarrow \quad 2\mathbf{a}\cdot\mathbf{c} = 1$

$\Rightarrow \quad |\mathbf{a}||\mathbf{c}|\cos\theta = \frac{1}{2}$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = \frac{\pi}{3}$

12 Since, position vectors of A, B, C are $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $a\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, respectively.

Now,

$\mathbf{AC} = (a\mathbf{i} - 3\mathbf{j} + \mathbf{k}) - (2\mathbf{i} - \mathbf{j} + \mathbf{k})$

$= (a - 2)\mathbf{i} - 2\mathbf{j}$

and $\mathbf{BC} = (a\mathbf{i} - 3\mathbf{j} + \mathbf{k}) - (\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$

$= (a - 1)\mathbf{i} + 6\mathbf{k}$

Since, the ΔABC is right angled at C, then

$\mathbf{BC}\cdot\mathbf{BC} = 0$

$\Rightarrow \quad \{(a - 2)\mathbf{i} - 2\mathbf{j}\}\cdot\{(a - 1)\mathbf{i} + 6\mathbf{k}\} = 0$

$\Rightarrow \quad (a - 2)(a - 1) = 0$

$\therefore \quad a = 1$ and $a = 2$

13 Since, the given vectors are mutually orthogonal, therefore

$\mathbf{a}\cdot\mathbf{b} = 2 - 4 + 2 = 0$

$\mathbf{a}\cdot\mathbf{c} = \lambda - 1 + 2\mu = 0 \quad$ ...(i)

$\mathbf{b}\cdot\mathbf{c} = 2\lambda + 4 + \mu = 0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$\mu = 2$ and $\lambda = -3$

Hence, $\quad (\lambda, \mu) = (-3, 2)$

14 Given that,

(i) $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, i.e. $|\mathbf{a}| = |\mathbf{b}| = 1$

(ii) $\mathbf{c} = \mathbf{a} + 2\mathbf{b}$ and $\mathbf{d} = 5\mathbf{a} - 4\mathbf{b}$

(iii) $\mathbf{c}$ and $\mathbf{d}$ are perpendicular to each other, i.e. $\mathbf{c}\cdot\mathbf{d} = 0$

Now, $\mathbf{c}\cdot\mathbf{d} = 0$

$\Rightarrow \quad (\mathbf{a} + 2\mathbf{b})\cdot(5\mathbf{a} - 4\mathbf{b}) = 0$

$\Rightarrow 5\mathbf{a}\cdot\mathbf{a} - 4\mathbf{a}\cdot\mathbf{b} + 10\mathbf{b}\cdot\mathbf{a} - 8\mathbf{b}\cdot\mathbf{b} = 0$

$\Rightarrow \quad 6\mathbf{a}\cdot\mathbf{b} = 3$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \frac{1}{2}$

So, the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$.

15 $\mathbf{p} \perp \mathbf{q}+\mathbf{r}, \mathbf{q} \perp \mathbf{r}+\mathbf{p}$ and $\mathbf{r} \perp \mathbf{p}+\mathbf{q}$

$\therefore \quad \mathbf{p} \cdot (\mathbf{q}+\mathbf{r}) = 0$

$\mathbf{q} \cdot (\mathbf{r}+\mathbf{p}) = 0$

$\mathbf{r} \cdot (\mathbf{p}+\mathbf{q}) = 0$

$\Rightarrow \quad \mathbf{p} \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{r} = 0$

$\mathbf{q} \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{p} = 0$

$\mathbf{r} \cdot \mathbf{p} + \mathbf{r} \cdot \mathbf{q} = 0$

On adding, we get

$$2\,(\mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{p}) = 0$$

Also, $\quad |\mathbf{p}+\mathbf{q}| = 6$

$\Rightarrow \quad |\mathbf{p}+\mathbf{q}|^2 = 36$

$\Rightarrow \quad \mathbf{p}^2 + \mathbf{q}^2 + 2\mathbf{p} \cdot \mathbf{q} = 36$

Similarly, $\mathbf{q}^2 + \mathbf{r}^2 + 2\mathbf{q} \cdot \mathbf{r} = 48$

and $\quad \mathbf{r}^2 + \mathbf{p}^2 + 2\,\mathbf{r} \cdot \mathbf{p} = 16$

Adding all, we get

$2(\mathbf{p}^2 + \mathbf{q}^2 + \mathbf{r}^2 + \mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{p}) = 100$

$\Rightarrow \quad 2\,(\mathbf{p}^2 + \mathbf{q}^2 + \mathbf{r}^2) = 100$

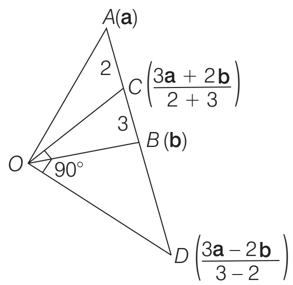
$[\because \mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{p} = 0]$

$\Rightarrow \quad \mathbf{p}^2 + \mathbf{q}^2 + \mathbf{r}^2 = 50$

$\Rightarrow \quad |\mathbf{p}+\mathbf{q}+\mathbf{r}|^2 = 50$

$|\mathbf{p}+\mathbf{q}+\mathbf{r}| = 5\sqrt{2}$

16. $\left(\dfrac{3\mathbf{a}+2\mathbf{b}}{5}\right) \cdot (3\mathbf{a} - 2\mathbf{b}) = 0$



$\Rightarrow \quad 9\,|\mathbf{a}|^2 - 4\,|\mathbf{b}|^2 = 0$

$\therefore \quad 9\mathbf{a}^2 = 4\mathbf{b}^2$

17 A vector coplanar with $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}+\mathbf{k}$ is $\mathbf{i}+\mathbf{j}+2\mathbf{k}+\lambda(\mathbf{i}+2\mathbf{j}+\mathbf{k})$

$= (1+\lambda)\mathbf{i} + (1+2\lambda)\mathbf{j} + (2+\lambda)\mathbf{k}$.

It is perpendicular to $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

$\therefore 1+\lambda+1+2\lambda+2+\lambda = 0 \Rightarrow \lambda = -1$

So, the required vector is $-\mathbf{j}+\mathbf{k}$.

18 Let first term and common ratio of a GP be α and β. Then,

$a = \alpha \cdot \beta^{p-1}, b = \alpha \cdot \beta^{q-1}, c = \alpha \cdot \beta^{r-1}$

$\therefore \log a = \log\alpha + (p-1)\log\beta$, etc.

The dot product of the given two vectors is

$\Sigma\{\log\alpha + (p-1)\log\beta\}\,(q-r)$

$= (\log\alpha - \log\beta)\,\Sigma\,(q-r)$

$+ \log\beta\,\Sigma\,p\,(q-r) = 0$

Hence, given vectors are perpendicular.

19 Here, $\mathbf{AB} = -3\mathbf{i}+\mathbf{k}$

Now, $\mathbf{AB} \cdot (2\mathbf{i}+3\mathbf{j}+6\mathbf{k}) = -6+6 = 0$

Hence, $\mathbf{AB}$ is perpendicular to the given line.

Thus, the required distance

$= |\mathbf{AB}| = \sqrt{9+1} = \sqrt{10}$

20 Given, $\mathbf{a} = 2\mathbf{i}-\mathbf{j}+\mathbf{k}, \mathbf{b} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{c} = \mathbf{i}+\mathbf{j}-2\mathbf{k}$

Now, we have;

$\mathbf{b}+\lambda\mathbf{c} = (1+\lambda)\mathbf{i} + (2+\lambda)\mathbf{j}$

$+ (-1-2\lambda)\mathbf{k}$

$\therefore$ Projection of $(\mathbf{b}+\lambda\mathbf{c})$ on

$\mathbf{a} = \left|\dfrac{(\mathbf{b}+\lambda\mathbf{c}) \cdot \mathbf{a}}{|\mathbf{a}|}\right| = \sqrt{\dfrac{2}{3}}$ [given]

$\Rightarrow \left|\dfrac{2(1+\lambda) - (2+\lambda) + (-1-2\lambda)}{\sqrt{4+1+1}}\right| = \sqrt{\dfrac{2}{3}}$

$\Rightarrow \left|\dfrac{-\lambda - 1}{\sqrt{6}}\right| = \dfrac{\sqrt{2}}{\sqrt{3}}$

$\Rightarrow \quad \lambda + 1 = 2 \quad \Rightarrow \quad \lambda = 1$

$\therefore \quad \mathbf{b}+\lambda\mathbf{c} = 2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$

21 We have, projection of $\mathbf{v}$ along $\mathbf{u}$ = Projection of $\mathbf{w}$ along $\mathbf{u}$

$\Rightarrow \quad \dfrac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|} = \dfrac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{u}|}$

$\Rightarrow \quad \mathbf{v} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u}$ ...(i)

Also, $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other.

$\therefore \quad \mathbf{v} \cdot \mathbf{w} = 0$ ...(ii)

Now,

$|\mathbf{u}-\mathbf{v}+\mathbf{w}| = |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2$

$-2\,(\mathbf{u} \cdot \mathbf{v}) - 2\,(\mathbf{v} \cdot \mathbf{w}) + 2\,(\mathbf{u} \cdot \mathbf{w})$

$\Rightarrow \quad |\mathbf{u}-\mathbf{v}+\mathbf{w}|^2 = 1+4+9$

[from Eqs. (i) and (ii)]

$\Rightarrow \quad |\mathbf{u}-\mathbf{v}+\mathbf{w}| = \sqrt{14}$

22 A vector parallel to the bisector of the angle between the vectors $\mathbf{a}$ and b is

$\dfrac{\mathbf{a}}{|\mathbf{a}|} + \dfrac{\mathbf{b}}{|\mathbf{b}|} = \mathbf{a}+\mathbf{b}$.

$\therefore$ Unit vector along the bisector

$= \dfrac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$

$= \dfrac{1}{\sqrt{2}}(\mathbf{a}+\mathbf{b})$

$\left[\begin{array}{l}\because |\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} \\ \Rightarrow |\mathbf{a}+\mathbf{b}|^2 = 1+1+0 = 2\end{array}\right]$

$\therefore$ Required projection

$= \left\{l \cdot \dfrac{\mathbf{a}}{|\mathbf{a}|} + m\,\dfrac{\mathbf{b}}{|\mathbf{a}|}\right.$

$\left. + n\,\dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}\right\} \cdot \dfrac{1}{\sqrt{2}}(\mathbf{a}+\mathbf{b})$

$= \dfrac{1}{\sqrt{2}}(l+m)$

$[\because |\mathbf{a}| = |\mathbf{b}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{a}\times\mathbf{b})$

$= \mathbf{b} \cdot (\mathbf{a}\times\mathbf{b}) = 0]$

23 $\mathbf{a}_1 = \dfrac{(\mathbf{a} \cdot \mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2}$

$\Rightarrow \quad \mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1 = \mathbf{a} - \dfrac{(\mathbf{a} \cdot \mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2}$

Thus, $\mathbf{a}_1 \times \mathbf{a}_2 = \dfrac{(\mathbf{a} \cdot \mathbf{b})\,\mathrm{b}}{|\mathbf{b}|^2} \times \left(\mathbf{a} - \dfrac{(\mathbf{a} \cdot \mathbf{b})\,\mathbf{b}}{|\mathbf{b}|^2}\right)$

$= \dfrac{(\mathbf{a} \cdot \mathbf{b})\,(\mathbf{b}\times\mathbf{a})}{|\mathbf{b}|^2}$

24 Total force,

$\mathbf{F} = (4\mathbf{i}+\mathbf{j}-3\mathbf{k}) + (3\mathbf{i}+\mathbf{j}-\mathbf{k})$

$\therefore \quad \mathbf{F} = 7\mathbf{i}+2\mathbf{j}-4\mathbf{k}$

The particle is displaced from

$A(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

to $\quad B(5\mathbf{i}+4\mathbf{j}+\mathbf{k})$.

Now, displacement,

$\mathbf{AB} = (5\mathbf{i}+4\mathbf{j}+\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

$= 4\mathbf{i}+2\mathbf{j}-2\mathbf{k}$

$\therefore$ Work done $= \mathbf{F} \cdot \mathbf{AB}$

$= (7\mathbf{i}+2\mathbf{j}-4\mathbf{k}) \cdot (4\mathbf{i}+2\mathbf{j}-2\mathbf{k})$

$= 28+4+8$

$= 40$ units

25 Since, $(2\mathbf{u}\times 3\mathbf{v})$ is a unit vector.

$\Rightarrow \quad |2\mathbf{u}\times 3\mathbf{v}| = 1$

$\Rightarrow \quad 6|\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta| = 1$

$\Rightarrow \quad \sin\theta = \dfrac{1}{6}$

$[\because |\mathbf{u}| = |\mathbf{v}| = 1]$

Since, θ is an acute angle, then there is exactly one value of θ for which $(2\mathbf{u}\times 3\mathbf{v})$ is a unit vector.

26 Since, the vectors $\mathbf{a} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $\mathbf{b} = \mathbf{j}$ are such that a, c and b form a right handed system.

$\therefore \quad \mathbf{c} = \mathbf{b}\times\mathbf{a} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ x & y & z\end{vmatrix}$

$= z\mathbf{i} - x\mathbf{k}$

27 Given, $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathrm{b} = 0\,\mathbf{i}+\mathbf{j}-\mathbf{k}$

Let $\quad \mathbf{c} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ such that $\mathbf{a}\times\mathbf{c} = \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c} = 3$

Now, $\mathbf{a}\times\mathbf{c} = \mathbf{b}$

$\Rightarrow \quad \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ x & y & z\end{vmatrix} = 0\,\mathbf{i}+\mathbf{j}-\mathbf{k}$

$\Rightarrow \quad (z-y)\,\mathbf{i} - \mathbf{j}\,(z-x) + \mathbf{k}\,(y-x)$

$= 0\,\mathbf{i}+\mathbf{j}-\mathbf{k}$

On comparing, we get

$z - y = 0 \Rightarrow y = z$ ...(i)

$-z + x = 1 \Rightarrow x = 1+z$ ...(ii)

and $\quad y - x = -1$ ...(iii)

Also, $\mathbf{a} \cdot \mathbf{c} = 3$

$\Rightarrow \quad (\mathbf{i}+\mathbf{j}+\mathbf{k}) \cdot (x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = 3$

$\Rightarrow \quad x + y + z = 3$ ...(iv)

On putting the values of x and y from Eqs. (i) and (ii) in Eq. (iv), we get

$(1+z) + z + z = 3$

$\Rightarrow \quad 3z = 2 \Rightarrow z = \frac{2}{3}$

On putting the value of z in Eqs. (i) and (ii), we get

$y = \frac{2}{3}$ and $x = \frac{5}{3}$

These values of x and y also satisfy Eq. (iii), we get

$x = \frac{5}{3}, y = \frac{2}{3}, z = \frac{2}{3}$

Hence, $\mathbf{c} = \frac{5}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, which is the required vector.

28 Given, $\mathbf{a} \cdot \mathbf{b} \neq 0, \mathbf{a} \cdot \mathbf{d} = 0$ ...(i)

and $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$

$\Rightarrow \mathbf{b} \times (\mathbf{c} - \mathbf{d}) = 0$

$\therefore \mathbf{b} \parallel (\mathbf{c} - \mathbf{d})$

$\Rightarrow \mathbf{c} - \mathbf{d} = \lambda \mathbf{b}$

$\Rightarrow \mathbf{d} = \mathbf{c} - \lambda \mathbf{d}$ ...(ii)

On taking dot product with $\mathbf{a}$, we get

$\mathbf{a} \cdot \mathbf{d} = \mathbf{a} \cdot \mathbf{c} - \lambda \mathbf{a} \cdot \mathbf{b}$

$\Rightarrow 0 = \mathbf{a} \cdot \mathbf{c} - \lambda (\mathbf{a} \cdot \mathbf{b})$ [from Eq. (i)]

$\Rightarrow \lambda = \frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}$ ...(iii)

$\therefore \mathbf{d} = \mathbf{c} - \frac{(\mathbf{a} \cdot \mathbf{c})}{(\mathbf{a} \cdot \mathbf{b})}\mathbf{b}$

29 Since, $[3\mathbf{u} \; p\mathbf{v} \; p\mathbf{w}] - [p\mathbf{v} \; \mathbf{w} \; q\mathbf{u}] - [2\mathbf{w} \; q\mathbf{v} \; q\mathbf{u}] = 0$

$\therefore 3p^2[\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})] - pq\,[\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})] - 2q^2\,[\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})] = 0$

$\Rightarrow (3p^2 - pq + 2q^2)\,[\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})] = 0$

But $[\mathbf{u} \; \mathbf{v} \; \mathbf{w}] \neq 0$

$\Rightarrow 3p^2 - pq + 2q^2 = 0$

$\therefore p = q = 0$

30 Here, $|\mathbf{u}| = 1$ and $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix}$

$= 3\mathbf{i} - 7\mathbf{j} - \mathbf{k}$

$|\mathbf{v} \times \mathbf{w}| = \sqrt{9 + 49 + 1} = \sqrt{59}$

$[\mathbf{u} \; \mathbf{v} \; \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \leq |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}| \leq \sqrt{59}$

31 Let $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$

$\mathbf{a} \cdot \mathbf{d} = (\mathbf{i} - \mathbf{j}) \cdot (d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k})$

$\Rightarrow d_1 - d_2 = 0$ $[\because \mathbf{a} \cdot \mathbf{d} = 0]$

$\Rightarrow d_1 = d_2$ ...(i)

Also, $\mathbf{d}$ is a unit vector.

$\Rightarrow d_1^2 + d_2^2 + d_3^2 = 1$ ...(ii)

Also, $[\mathbf{b} \; \mathbf{c} \; \mathbf{d}] = 0$

$\Rightarrow \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

$\Rightarrow -1(-d_3 - d_1) - 1(-d_2) = 0$

$\Rightarrow d_1 + d_2 + d_3 = 0$

$\Rightarrow 2d_1 + d_3 = 0$ [from Eq. (i)]

$\Rightarrow d_3 = -2d_1$ ...(iii)

Using Eqs. (iii) and (i) in Eq. (ii), we get

$d_1^2 + d_1^2 + 4d_1^2 = 1 \Rightarrow d_1 = \pm \frac{1}{\sqrt{6}}$

$\therefore d_2 = \pm \frac{1}{\sqrt{6}}$

and $d_3 = \mp \frac{2}{\sqrt{6}}$

Hence, required vector is

$\pm \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.

32 $(\mathbf{i} \times \mathbf{a} \cdot \mathbf{b})\mathbf{i} + (\mathbf{j} \times \mathbf{a} \cdot \mathbf{b})\mathbf{j} + (\mathbf{k} \times \mathbf{a} \cdot \mathbf{b})\mathbf{k}$

$= [\mathbf{i} \; \mathbf{a} \; \mathbf{b}]\mathbf{i} + [\mathbf{j} \; \mathbf{a} \; \mathbf{b}]\mathbf{j} + [\mathbf{k} \; \mathbf{a} \; \mathbf{b}]\mathbf{k}$

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$\therefore [\mathbf{iab}] = \begin{vmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - b_2a_3)$

$[\mathbf{j} \, \mathbf{ab}] = \begin{vmatrix} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (b_1a_3 - a_1b_3)$

and $[\mathbf{k} \, \mathbf{ab}] = \begin{vmatrix} 0 & 0 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= (a_1b_2 - a_2b_1)$

$\therefore [\mathbf{i} \; \mathbf{a} \; \mathbf{b}]\mathbf{i} + [\mathbf{j} \; \mathbf{a} \; \mathbf{b}]\mathbf{j} + [\mathbf{k} \; \mathbf{a} \; \mathbf{b}]\mathbf{k}$

$= (a_2b_3 - b_2a_3)\mathbf{i} + (b_1a_3 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} = \mathbf{a} \times \mathbf{b}$

33 Let P, Q, R and S be the given points with position vectors $\alpha\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \beta\mathbf{k}$ respectively. Then,

$\mathbf{QP} = (\alpha - 1)\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$,

$\mathbf{QR} = 0\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$

and $\mathbf{QS} = 0\mathbf{i} + 2\mathbf{j} + (\beta + 1)\mathbf{k}$ are coplanar

$\therefore [\mathbf{QP} \; \mathbf{QR} \; \mathbf{QS}] = 0$

$\Rightarrow \begin{vmatrix} \alpha - 1 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 2 & \beta + 1 \end{vmatrix} = 0$

$\Rightarrow (\alpha - 1)(\beta + 1) = 0$

$\Rightarrow (1 - \alpha)(1 + \beta) = 0$

34 Since, given vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

$\therefore \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x - 2 & -1 \end{vmatrix} = 0$

$\Rightarrow 1\{1 - 2(x - 2)\} - 1(-1 - 2x) + 1(x - 2 + x) = 0$

$\Rightarrow 1 - 2x + 4 + 1 + 2x + 2x - 2 = 0$

$\Rightarrow 2x = -4$

$\Rightarrow x = -2$

35 Given, $\mathbf{a} = p\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + q\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} + r\mathbf{k}$ are coplanar and $p \neq q \neq r \neq 1$.

Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

$\Rightarrow [\mathbf{a} \; \mathbf{b} \; \mathbf{c}] = 0$

$\Rightarrow \begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$

$\Rightarrow p(qr - 1) - 1(r - 1) + 1(1 - q) = 0$

$\Rightarrow pqr - p - r + 1 + 1 - q = 0$

$\therefore pqr - (p + q + r) = -2$

36 It is given that α, β and γ are coplanar vectors.

$\therefore [\alpha \; \beta \; \gamma] = 0$

$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

$\Rightarrow 3abc - a^3 - b^3 - c^3 = 0$

$\Rightarrow a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow a + b + c = 0$

$[\because a^2 + b^2 + c^2 - ab - bc - ca \neq 0]$

$\Rightarrow \mathbf{v} \cdot \alpha = \mathbf{v} \cdot \beta = \mathbf{v} \cdot \gamma = 0$

Hence, $\mathbf{v}$ is perpendicular to α, β and γ.

37 Given that,

$[\lambda(\mathrm{a} + \mathrm{b}) \;\; \lambda^2\mathrm{b} \;\; \lambda\mathrm{c}] = [\mathrm{a} \;\; \mathrm{b} + \mathrm{c} \;\; \mathrm{b}]$

$\therefore \begin{vmatrix} \lambda(a_1 + b_1) & \lambda(a_2 + b_2) & \lambda(a_3 + b_3) \\ \lambda^2 b_1 & \lambda^2 b_2 & \lambda^2 b_3 \\ \lambda c_1 & \lambda c_2 & \lambda c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$\Rightarrow \lambda^4 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\Rightarrow \lambda^4 = -1$

So, no real value of λ exists.

38 As $\mathbf{a} + 3\mathbf{b}$ is collinear with $\mathbf{c}$.

$\Rightarrow \mathbf{a} + 3\mathbf{b} = \lambda\mathbf{c}$ ...(i)

Also, $\mathbf{b} + 2\mathbf{c}$ is collinear with $\mathbf{a}$.

$\Rightarrow \mathbf{b} + 2\mathbf{c} = \mu\,\mathbf{a}$ ...(ii)

From Eq. (i),

$\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (\lambda + 6)\mathbf{c}$ ...(iii)

From Eq. (ii),

$\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (1 + 3\mu)\mathbf{a}$ ...(iv)

From Eqs. (iii) and (iv), we get

$(\lambda + 6)\mathbf{c} = (1 + 3\mu)\mathbf{a}$

Since, $\mathbf{a}$ is not collinear with $\mathbf{c}$.

$\Rightarrow \lambda + 6 = 1 + 3\mu = 0$

From Eq. (iv), $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = 0$

39 We know, $[\mathbf{a} \times \mathbf{b} \;\; \mathbf{b} \times \mathbf{c} \;\; \mathbf{c} \times \mathbf{a}] = [\mathbf{a} \; \mathbf{b} \; \mathbf{c}]^2$

$\therefore \lambda = 1$

40. We have, $|[\mathbf{a} \; \mathbf{b} \, \mathbf{c}]| = V$

Let V_1 be the volume of the parallelopiped formed by the vectors α, β and γ.

Then, $V_1 = |[\alpha \; \beta \; \gamma]|$

Now, $[\alpha\ \beta\ \gamma] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{c} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\Rightarrow \quad [\alpha\ \beta\ \gamma] = [\mathbf{abc}]^2\ [\mathbf{abc}]$

$\Rightarrow \quad [\alpha\ \beta\ \gamma] = [\mathbf{abc}]^3$

$\therefore \quad V_1 = |[\alpha\ \beta\ \gamma]|$

$= |[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^3| = V^3$

41 Let $\alpha = \mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, \beta = \lambda\mathbf{b} + 4\mathbf{c}$ and $\gamma = (2\lambda - 1)\mathbf{c}$

Then, $[\alpha\ \beta\ \gamma] = \begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & (2\lambda - 1) \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\Rightarrow \quad [\alpha\ \beta\ \gamma] = \lambda(2\lambda - 1)[\mathbf{abc}]$

$\Rightarrow \quad [\alpha\ \beta\ \gamma] = 0$, if $\lambda = 0, \frac{1}{2}$ $\quad [\because [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0]$

Hence, α, β and γ are non-coplanar for all values of λ except two values 0 and $\frac{1}{2}$.

42 Given, $\quad \mathbf{a} = \frac{1}{\sqrt{10}}(3\mathbf{i} + \mathbf{k})$

and $\quad \mathbf{b} = \frac{1}{7}(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$

$\therefore\ (2\mathbf{a} - \mathbf{b})\cdot\{(\mathbf{a}\times\mathbf{b})\times(\mathbf{a} + 2\mathbf{b})\}$

$= (2\mathbf{a} - \mathbf{b})\cdot\{(\mathbf{a}\times\mathbf{b})\times\mathbf{a} + (\mathbf{a}\times\mathbf{b})\times 2\mathbf{b}\}$

$= (2\mathbf{a} - \mathbf{b})\cdot\{(\mathbf{a}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{a} + 2(\mathbf{a}\cdot\mathbf{b})\mathbf{b} - 2(\mathbf{b}\cdot\mathbf{b})\mathbf{a}\}$

$= (2\mathbf{a} - \mathbf{b})\cdot\{1(\mathbf{b}) - (0)\mathbf{a} + 2(0)\mathbf{b} - 2(1)\mathbf{a}\}$

$[\mathbf{a}\cdot\mathbf{b} = 0$ and $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = 1]$

$= (2\mathbf{a} - \mathbf{b})(\mathbf{b} - 2\mathbf{a})$

$= -(4|\mathbf{a}|^2 - 4\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2)$

$= -\{4 - 0 + 1\} = -5$

43 Given $|\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = |\hat{\mathbf{c}}| = 1$

and $\hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}}) = \frac{\sqrt{3}}{2}(\hat{\mathbf{b}} + \hat{\mathbf{c}})$

Now, consider

$\hat{\mathbf{a}}\times(\hat{\mathbf{b}}\times\hat{\mathbf{c}}) = \frac{\sqrt{3}}{2}(\hat{\mathbf{b}} + \hat{\mathbf{c}})$

$\Rightarrow \quad (\hat{\mathbf{a}}\cdot\hat{\mathbf{c}})\hat{\mathbf{b}} - (\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})\hat{\mathbf{c}} = \frac{\sqrt{3}}{2}\hat{\mathbf{b}} + \frac{\sqrt{3}}{2}\hat{\mathbf{c}}$

On comparing, we get

$\hat{\mathbf{a}}\cdot\hat{\mathbf{b}} = -\frac{\sqrt{3}}{2} \Rightarrow |\hat{\mathbf{a}}||\hat{\mathbf{b}}|\cos\theta = -\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \cos\theta = -\frac{\sqrt{3}}{2} \quad [\because |\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1]$

$\Rightarrow \quad \cos\theta = \cos\left(\pi - \frac{\pi}{6}\right) \Rightarrow \theta = \frac{5\pi}{6}$

44 We have, $\mathbf{a}\times\mathbf{b} + \mathbf{c} = 0$

$\Rightarrow \quad \mathbf{a}\times(\mathbf{a}\times\mathbf{b}) + \mathbf{a}\times\mathbf{c} = 0$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b} + \mathbf{a}\times\mathbf{c} = 0$

$\Rightarrow \quad 3\mathbf{a} - 2\mathbf{b} + \mathbf{a}\times\mathbf{c} = 0$

$\Rightarrow \quad 2\mathbf{b} = 3\mathbf{a} + \mathbf{a}\times\mathbf{c}$

$\Rightarrow \quad 2\mathbf{b} = 3\mathbf{j} - 3\mathbf{k} - 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

$= -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

$\therefore \quad \mathbf{b} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

45 Key Idea If any vector $\mathbf{x}$ is coplanar with the vector $\mathbf{y}$ and $\mathbf{z}$, then

$\mathbf{x} = \lambda\mathbf{y} + \mu\mathbf{b}$

Here, $\mathbf{u}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$

$\therefore \quad \mathbf{u} = \lambda\mathbf{a} + \mu\mathbf{b}$

Dot product with $\mathbf{a}$, we get ... (i)

$[\because \mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}, \mathbf{b} = \mathbf{j} + \mathbf{k}, \mathbf{u}\cdot\mathbf{a} = \mathbf{0}]$

Dot product with $\mathbf{b}$, we get

$\mathbf{u}\cdot\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b}) + \mu(\mathbf{b}\cdot\mathbf{b})$

$24 = 2\lambda + 2\mu$ ... (ii)

$[\because \mathbf{u}\cdot\mathbf{b} = 24]$

Solving Eqs. (i) and (ii), we get

$\lambda = -2, \mu = 14$

Dot product with $\mathbf{u}$, we get

$|\mathbf{u}|^2 = \lambda(\mathbf{u}\cdot\mathbf{a}) + \mu(\mathbf{u}\cdot\mathbf{b})$

$|\mathbf{u}|^2 = -2(0) + 14(24)$

$\Rightarrow \quad |\mathbf{u}|^2 = 336$

46 Since, $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} = 0$ and $\mathbf{b}\cdot\mathbf{c} = \frac{1}{2}$

$\therefore \quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}\times\mathbf{c}| = 1$

Now, $|\mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c}|^2$

$= |\mathbf{a}\times\mathbf{b}|^2 + |\mathbf{a}\times\mathbf{c}|^2 - 2(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})$

$= 1 + 1 - 2\begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = 1$

47 Since, $\mathbf{d}$ is collinear to vector $\mathbf{c}$.

$\therefore \quad \mathbf{c} = \lambda\mathbf{d}$

$\Rightarrow \quad (2x + 3)\mathbf{a} + \mathbf{b} = \lambda[(2x + 3)\mathbf{a} - \mathbf{b}]$

$\Rightarrow \quad 2x\mathbf{a} + 3\mathbf{a} + \mathbf{b} = 2\lambda x\mathbf{a} + 3\lambda\mathbf{a} - \mathbf{b}\lambda$

$\Rightarrow \quad (2x - 2\lambda x\mathbf{a} + 3\mathbf{a} - 3\lambda\mathbf{a}) + (\mathbf{b}\lambda + \mathbf{b}) = 0$

$\Rightarrow (2x - 2\lambda x + 3 - 3\lambda)\mathbf{a} + (\lambda + 1)\mathbf{b} = 0$

$\therefore \quad (2x - 2\lambda x + 3 - 3\lambda) = 0$ and $(\lambda + 1) = 0$

$\Rightarrow (2x + 3) - \lambda(2x + 3) = 0$ and $\lambda = -1$

$\Rightarrow \quad (2x + 3)(1 - \lambda) = 0$

$\therefore \quad x = -\frac{3}{2}$ and $\lambda = 1$

48. We have, $\quad \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$

and $\quad |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$

Let vector $(\mathbf{a} + \mathbf{b} + \mathbf{c})$ be inclined to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ at angles α, β and γ, respectively.

Then,

$\cos\alpha = \frac{(\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot\mathbf{a}}{|\mathbf{a} + \mathbf{b} + \mathbf{c}||\mathbf{a}|}$

$= \frac{\mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{a} + \mathbf{c}\cdot\mathbf{a}}{|\mathbf{a} + \mathbf{b} + \mathbf{c}||\mathbf{a}|}$

$= \frac{|\mathbf{a}|}{|\mathbf{a} + \mathbf{b} + \mathbf{c}|}$

Similarly, $\cos\beta = \frac{|\mathbf{b}|}{|\mathbf{a} + \mathbf{b} + \mathbf{c}|}$

and $\cos\gamma = \frac{|\mathbf{c}|}{|\mathbf{a} + \mathbf{b} + \mathbf{c}|}$

Now, as $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$, therefore $\cos\alpha = \cos\beta = \cos\gamma$

$\therefore \quad \alpha = \beta = \gamma$

Hence, the vector $(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is equally inclined to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

49 $\quad V = \begin{vmatrix} 1 & a & 0 \\ a & 1 & 1 \\ 0 & 1 & a \end{vmatrix} = a - 1 - a^3$

$\therefore \quad \frac{dV}{da} = 1 - 3a^2 = 0$ (say)

Now, $a = \pm\frac{1}{\sqrt{3}}$ and $\frac{d^2V}{da^2} = -6a$

$\Rightarrow \quad \left(\frac{d^2V}{da^2}\right)_{\left(a = \frac{1}{\sqrt{3}}\right)} = -\frac{6}{\sqrt{3}}$ (– ve)

Hence, V is maximum at $a = \frac{1}{\sqrt{3}}$.

50 Since, $\mathbf{b} = \mathbf{r}\times\mathbf{a}$

We have, $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times(\mathbf{r}\times\mathbf{a})$

$= (\mathbf{a}\cdot\mathbf{a})\mathbf{r} - (\mathbf{a}\cdot\mathbf{r})\mathbf{a}$

$= (\mathbf{a}\cdot\mathbf{a})\mathbf{r}$

$\because \quad \mathbf{a}\cdot\mathbf{r} = 0$

$\therefore \quad \mathbf{r} = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}$

SESSION 2

1 We have, $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

$\Rightarrow \quad |\mathbf{a}| = \sqrt{4 + 1 + 4} = 3$

and $\quad \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} \Rightarrow |\mathbf{b}|\sqrt{1 + 1} = \sqrt{2}$

Now, $\quad |\mathbf{c} - \mathbf{a}| = 3 \Rightarrow |\mathbf{c} - \mathbf{a}|^2 = 9$

$\Rightarrow \quad (\mathbf{c} - \mathbf{a})\cdot(\mathbf{c} - \mathbf{a}) = 9$

$\Rightarrow \quad |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{c}\cdot\mathbf{a} = 9$ ... (i)

Again, $\quad |(\mathbf{a}\times\mathbf{b})\times\mathbf{c}| = 3$

$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}||\mathbf{c}|\sin 30° = 3$

$\Rightarrow \quad |\mathbf{c}| = \frac{6}{|\mathbf{a}\times\mathbf{b}|}$

But $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

$\therefore \quad |\mathbf{c}| = \frac{6}{\sqrt{4 + 4 + 1}} = 2$ ... (ii)

From Eqs. (i) and (ii), we get

$(2)^2 + (3)^2 - 2\mathbf{c}\cdot\mathbf{a} = 9$

$\Rightarrow \quad 4 + 9 - 2\mathbf{c}\cdot\mathbf{a} = 9 \Rightarrow \mathbf{c}\cdot\mathbf{a} = 2$

2 Given two vectors lie in xy-plane. So, a vector coplanar with them is

$\mathbf{a} = x\mathbf{i} + y\mathbf{j}$

Since, $\quad \mathbf{a} \perp (\mathbf{i} - \mathbf{j})$

$\Rightarrow \quad (x\mathbf{i} + y\mathbf{j})\cdot(\mathbf{i} - \mathbf{j}) = 0$

$\Rightarrow \quad x - y = 0$

$\Rightarrow \quad x = y$

$\therefore \quad \mathbf{a} = x\mathbf{i} + x\mathbf{j}$

and $\quad |\mathbf{a}| = \sqrt{x^2 + x^2} = x\sqrt{2}$

∴ Required unit vector

$$= \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{x(\mathbf{i} + \mathbf{j})}{x\sqrt{2}} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$$

3 Now, $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$

$= 16 - 4 = 12$

and $|\mathbf{c}|^2 = (2\mathbf{a} \times \mathbf{b} - 3\mathbf{b}) \cdot (2\mathbf{a} \times \mathbf{b} - 3\mathbf{b})$

$= 4|\mathbf{a} \times \mathbf{b}|^2 + 9|\mathbf{b}|^2$

$= 4 \cdot 12 + 9 \cdot 16$

$= 192$

$\Rightarrow \quad |\mathbf{c}| = 8\sqrt{3}$

Now, $\mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot (2\mathbf{a} \times \mathbf{b} - 3\mathbf{b})$

$= -3|\mathbf{b}|^2 = -48$

$\because \quad \cos\theta = \dfrac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}||\mathbf{c}|}$

$= -\dfrac{48}{4 \cdot 8\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$

$\therefore \quad \theta = \dfrac{5\pi}{6}$

4 Let $\mathbf{d} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \cos\alpha$

$\Rightarrow \mathbf{d} \cdot (\mathbf{a} - \mathbf{k}) = 0$ and $\mathbf{d} \cdot (\mathbf{b} - \mathbf{k}) = 0$

$\mathbf{d}$ is parallel to $(\mathbf{a} - \mathbf{k}) \times (\mathbf{b} - \mathbf{k})$

$$\therefore \quad \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -1 \\ -\sin\theta & \cos\theta & -1 \end{vmatrix}$$

$$\Rightarrow \quad \mathbf{d} = \frac{(\cos\theta - \sin\theta)\,\mathbf{i} + (\cos\theta + \sin\theta)\,\mathbf{j} + \mathbf{k}}{\sqrt{3}}$$

$\cos\alpha = \mathbf{d} \cdot \mathbf{k} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

5 Hence, $\mathbf{p} \times \mathbf{q} = \{3ax^3 + 2b(x-1)^2\}\mathbf{k} = f(x)\,\mathbf{k}$,

where, $f(0)\,f(1) = 6ab < 0$

∴ By intermediate value theorem there exists, x in $(0, 1)$ such that $f(x) = 0$.

6
$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix} = \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$= 1 - 3\cos^2\theta + 2\cos^3\theta$

$= (1 - \cos\theta)^2 (1 + 2\cos\theta)$

$\Rightarrow \quad 1 + 2\cos\theta \geq 0 \Rightarrow \theta \leq \dfrac{2\pi}{3}$

7 Let $\mathbf{c} = x\mathbf{i} + y\mathbf{j}$

Then, $\mathbf{b} \perp \mathbf{c} \Rightarrow \mathbf{b} \cdot \mathbf{c} = 0 \Rightarrow 4x + 3y = 0$

$\Rightarrow \quad \dfrac{x}{3} = \dfrac{y}{-4} = \lambda$ [say]

$\Rightarrow \quad x = 3\lambda,\ y = -4\lambda$

$\therefore \quad \mathbf{c} = \lambda(3\mathbf{i} - 4\mathbf{j})$

Let the required vector be $\alpha = p\mathbf{i} + q\mathbf{j}$.

Then, the projections of α on $\mathbf{b}$ and $\mathbf{c}$ are $\dfrac{\alpha \cdot \mathbf{b}}{|\mathbf{b}|}$ and $\dfrac{\alpha \cdot \mathbf{c}}{|\mathbf{c}|}$, respectively.

$\therefore \quad \alpha \cdot \dfrac{\mathbf{b}}{|\mathbf{b}|} = 1$ and $\dfrac{\alpha \cdot \mathbf{c}}{|\mathbf{c}|} = 2$ [given]

$\Rightarrow \quad 4p + 3q = 5$ and $3p - 4q = 10$

$\Rightarrow \quad p = 2, q = -1$

$\alpha = 2\mathbf{i} - \mathbf{j}$

8 A vector coplanar to $(2\mathbf{i} + \mathbf{j} + \mathbf{k})$, $(\mathbf{i} - \mathbf{j} + \mathbf{k})$ and orthogonal to $(3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$

$= \lambda\,[\{(2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k})\} \times (3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})]$

$= \lambda\,(21\mathbf{j} - 7\mathbf{k})$

∴ Required unit vector is

$$\pm \frac{(21\mathbf{j} - 7\mathbf{k})}{\sqrt{(21)^2 + (7)^2}} = \pm \frac{(3\mathbf{j} - \mathbf{k})}{\sqrt{10}}$$

9 Let $\mathbf{a} = 2x^2\,\mathbf{i} + 4x\,\mathbf{j} + \mathbf{k}$

and $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + x\mathbf{k}$.

The angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.

$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} < 0 \quad \Rightarrow \quad 14x^2 - 8x + x < 0$

$\Rightarrow \quad 7x(2x - 1) < 0$

$\therefore \quad x \in \left(0, \dfrac{1}{2}\right)$ ...(i)

Also, it is given, $\mathbf{b} \cdot \mathbf{k} = x$

and $\dfrac{\mathbf{b} \cdot \mathbf{k}}{|\mathbf{b}|} < \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad 2x > \sqrt{3}\sqrt{53 + x^2}$

$\therefore \quad x^2 > 159$ ...(ii)

Hence, there is no common value for Eqs. (i) and (ii).

10 Let θ be an angle between unit vectors $\mathbf{a}$ and $\mathbf{b}$.

Then, $\mathbf{a} \cdot \mathbf{b} = \cos\theta$

Now, $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}$

$= 2 + 2\cos\theta = 4\cos^2\dfrac{\theta}{2}$

$\Rightarrow |\mathbf{a} + \mathbf{b}| = 2\cos\dfrac{\theta}{2}$

Similarly, $|\mathbf{a} - \mathbf{b}| = 2\sin\dfrac{\theta}{2}$

$\Rightarrow |\mathbf{a} + \mathbf{b}| + |\mathbf{a} - \mathbf{b}|$

$= 2\left(\cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}\right) \leq 2\sqrt{2}$

11 We have, $\mathbf{u} \cdot \mathbf{n} = 0$ and $\mathbf{v} \cdot \mathbf{n} = 0$

$\Rightarrow \quad \mathbf{n} \perp \mathbf{u}$ and $\mathbf{n} \perp \mathbf{v}$

$\Rightarrow \quad \mathbf{n} = \pm \dfrac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}$

Now, $\mathbf{u} \times \mathbf{v} = (\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j}) = -2\mathbf{k}$

$\therefore \quad \mathbf{n} = \pm\mathbf{k}$

Hence,

$|\mathbf{w} \cdot \mathbf{n}| = |(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (\pm\mathbf{k})| = 3$

12 By expanding $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, we get

$\mathbf{a} \cdot \mathbf{c} = x^2 - 2x + 6,\ \mathbf{a} \cdot \mathbf{b} = -\sin y$

Given, $\quad \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 4$

$\Rightarrow \quad x^2 - 2x + 2 = \sin y$

$\Rightarrow \quad \sin y = x^2 - 2x + 2$

$= (x - 1)^2 + 1 \geq 1$

But $\sin y \leq 1$

So, both sides are equal only for $x = 1$.

13 If $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{0}$, then $\mathbf{c} = \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$,

$\therefore \quad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = |\mathbf{a} \times \mathbf{b}|$

$$\Rightarrow \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = |\mathbf{a} \times \mathbf{b}|^2$$

14 Here, $\mathbf{a} \cdot \mathbf{x} = 2$ and $\mathbf{a} \times \mathbf{r} + \mathbf{b} = \mathbf{r}$ ...(i)

Dot product of Eq. (i) with $\mathbf{a}$ gives,

$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{r} = 2$

Cross product of Eq. (i) with $\mathbf{a}$ gives

$\mathbf{a} \times (\mathbf{a} \times \mathbf{r}) + \mathbf{a} \times \mathbf{b}$

$= \mathbf{a} \times \mathbf{r} = \mathbf{r} - \mathbf{b}$ [from Eq. (i)]

$\Rightarrow \quad 2\mathbf{a} - \mathbf{r} + \mathbf{a} \times \mathbf{b} = \mathbf{r} - \mathbf{b}$

$\therefore \quad \mathbf{r} = \dfrac{1}{2}[2\mathbf{a} + \mathbf{b} + \mathbf{a} \times \mathbf{b}]$

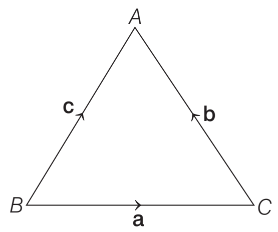
15 Since, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

Taking cross product with $\mathbf{a}$, we get

$\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = 0$ or $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$

Similarly, $\quad \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b}$

Thus $\quad \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$



Now, as $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors and $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, therefore $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ represents an equilateral triangle.

Hence, $\quad \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \neq 0$

16 Taking O as the origin, let the position vectors of A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively.

Then, the position vectors G_1, G_2 and G_3 are $\dfrac{\mathbf{b} + \mathbf{c}}{3}$, $\dfrac{\mathbf{c} + \mathbf{a}}{3}$

and $\dfrac{\mathbf{a} + \mathbf{b}}{3}$, respectively.

$\therefore \quad V_1 = \dfrac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

and $\quad V_2 = [\mathbf{OG}_1\ \mathbf{OG}_2\ \mathbf{OG}_3]$

Now, $\quad V_2 = [\mathbf{OG}_1, \mathbf{OG}_2, \mathbf{OG}_3]$

$$= \left[\frac{\mathbf{b} + \mathbf{c}}{3}\ \frac{\mathbf{c} + \mathbf{a}}{3}\ \frac{\mathbf{a} + \mathbf{b}}{3}\right]$$

$$= \frac{1}{27}[\mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}\ \ \mathbf{a}+\mathbf{b}]$$

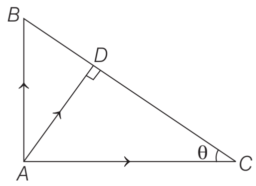
$$= \frac{2}{27}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \frac{2}{27} \times 6V_1$$

$$\Rightarrow \quad 9V_2 = 4V_1$$

17 Let $|\mathbf{BC}| = l$

In ΔABC, $l = \sqrt{AB^2 + AC^2}$

$$\therefore \quad \tan\theta = \frac{AB}{AC}$$



$$\Rightarrow \sin\theta = \frac{AB}{l} \text{ and } \cos\theta = \frac{AC}{l}$$

$$\therefore \quad \text{Resultant vector} = \frac{1}{AB}\mathbf{i} + \frac{1}{AC}\mathbf{j}$$

$$= \left(\frac{1}{l\sin\theta}\mathbf{i} + \frac{1}{l\cos\theta}\mathbf{j}\right)$$

In ΔADC,

$$AD = AC\sin\theta = l\sin\theta\cos\theta$$

$$= \frac{AB \cdot AC}{l}$$

$\therefore$ Magnitude of resultant vector

$$= \sqrt{\frac{1}{l^2}\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}\right)}$$

$$= \frac{l}{(AB)(AC)} = \frac{1}{AD}$$

18 Given, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$

$$\Rightarrow \quad -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\Rightarrow -(\mathbf{c}\cdot\mathbf{b})\cdot\mathbf{a} + (\mathbf{c}\cdot\mathbf{a})\mathbf{b} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$$

$$\left[\frac{1}{3}|\mathbf{b}||\mathbf{c}| + (\mathbf{c}\cdot\mathbf{b})\right]\mathbf{a} = (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}$$

Since, $\mathbf{a}$ and $\mathbf{b}$ are not collinear.

$$\mathbf{c}\cdot\mathbf{b} + \frac{1}{3}|\mathbf{b}||\mathbf{c}| = 0 \text{ and } \mathbf{c}\cdot\mathbf{a} = 0$$

$$\Rightarrow |\mathbf{c}||\mathbf{b}|\cos\theta + \frac{1}{3}|\mathbf{b}||\mathbf{c}| = 0$$

$$\Rightarrow \quad |\mathbf{b}||\mathbf{c}|\left(\cos\theta + \frac{1}{3}\right) = 0$$

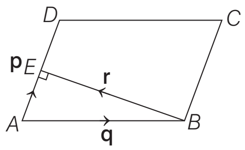
$$\Rightarrow \quad \cos\theta + \frac{1}{3} = 0$$

$$[\because |\mathbf{b}| \neq 0, |\mathbf{c}| \neq 0]$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{3}$$

$$\Rightarrow \quad \sin\theta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

19 Given,



(i) A parallelogram $ABCD$ such that $\mathbf{AB} = \mathbf{q}$ and $\mathbf{AD} = \mathbf{p}$.

(ii) The altitude from vertex B to side AD coincides with a vector $\mathbf{r}$.

To find The vector $\mathbf{r}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

Let E be the foot of perpendicular from B to side AD.

AE = Projection of vector $\mathbf{q}$

$\mathbf{AE}$ = Vector along AE of length AE

$$= |\mathbf{AE}|\,\mathbf{AE}$$

$$= \frac{(\mathbf{q}\cdot\mathbf{p})\,\mathbf{p}}{|\mathbf{p}|^2}$$

Now, applying triangles law in ΔABE, we get

$$\mathbf{AB} + \mathbf{BE} = \mathbf{AE}$$

$$\Rightarrow \quad \mathbf{q} + \mathbf{r} = \frac{(\mathbf{q}\cdot\mathbf{p})\,\mathbf{p}}{|\mathbf{p}|^2}$$

$$\Rightarrow \quad \mathbf{r} = \frac{(\mathbf{q}\cdot\mathbf{p})\,\mathbf{p}}{|\mathbf{p}|^2} - \mathbf{q}$$

$$= -\mathbf{q} + \left(\frac{\mathbf{q}\cdot\mathbf{p}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$$

20 In an isosceles ΔABC in which $AB = AC$, the median and bisector from A must be same line. Statement II is true.

Now, $\quad \mathbf{AD} = \frac{\mathbf{u}+\mathbf{v}}{2}$

and $\quad |\mathbf{AD}|^2 = \frac{1}{2}\cdot 2\cos^2\frac{\alpha}{2}$

So, $\quad |\mathbf{AD}| = \cos\frac{\alpha}{2}$

Unit vector along AD, i.e. x is given by

$$\mathbf{x} = \frac{\mathbf{AD}}{|\mathbf{AD}|} = \frac{\mathbf{u}+\mathbf{v}}{2\cos\frac{\alpha}{2}}.$$